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A MULTIGRID METHOD FOR NONLOCAL PROBLEMS: NON–DIAGONALLY DOMINANT OR TOEPLITZ-PLUS-TRIDIAGONAL SYSTEMS*

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Abstract. Nonlocal problems have been used to model very different applied scientific phenomena which involve the fractional Laplacian when one looks at the Lévy processes and stochastic interfaces. This paper deals with the nonlocal problems on a bounded domain where the stiffness matrices of the resulting systems are Toeplitz plus tridiagonal or far from being diagonally dominant as occurs when dealing with linear finite element approximations. By exploiting a weakly diagonally dominant Toeplitz property of the stiffness matrices, the optimal convergence of the two-grid method is well established in [Fiorentino and Serra-Capizzano, *SIAM J. Sci. Comput.*, 17 (1996), pp. 1068–1081; Chen and Deng, *SIAM J. Matrix Anal. Appl.*, 38 (2017), pp. 869–890], and there are still questions about the best ways to define the coarsening and interpolation operators when the stiffness matrix is far from being weakly diagonally dominant [Stüben, *J. Comput. Appl. Math.*, 128 (2001), pp. 281–309]. In this work, using spectral indications from our analysis of the involved matrices, the simple (traditional) restriction operator and prolongation operator are employed in order to handle general algebraic systems which are *neither Toeplitz nor weakly diagonally dominant* corresponding to the fractional Laplacian kernel and the constant kernel, respectively. We focus our efforts on providing the detailed proof of the convergence of the two-grid method for such situations. Moreover, the convergence of the full multigrid is also discussed with the constant kernel. The numerical experiments are performed to verify the convergence with only $\mathcal{O}(N\log N)$ complexity by the fast Fourier transform, where N is the number of the grid points.

Key words. multigrid methods, nonlocal problems, Toeplitz-plus-tridiagonal system, non–diagonally dominant system, fast Fourier transform

AMS subject classifications. 26A33, 65M55, 65T50

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1. Introduction.

Nonlocal diffusion problems have been used to model very different scientific phenomena occurring in various applied fields, for example, in biology, particle systems, image processing, coagulation models, mathematical finance, etc. [3]. When one looks at the Lévy processes and stochastic interfaces, the nonlocal operator that appears naturally is the fractional Laplacian [3, 8, 48]. Recently, the nonlocal volume-constrained diffusion problems, the so-called nonlocal model for distinguishing the nonlocal diffusion problems, attracted the wide interest of scientists [2, 23], where the linear scalar peridynamic model can be considered as a special case [23, 41]. In particular, it is pointed out that the fractional Laplacian for anomalous diffusion is a special case of the nonlocal model [21, 23]. In the literature, we have already a lot of important proposals for numerically solving nonlocal problems or

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nonlocal models. For example, a finite difference-quadrature scheme for the fractional Laplacian has been derived in [27]. Finite element approximations for the fractional Laplacian [1, 21] or tempered fractional Laplacian have been discussed in [47]. A fast conjugate gradient Galerkin method has been used for solving efficiently the resulting system arising from a peridynamic model [44]. In this work, we especially focus our efforts on the efficient multigrid method (MGM) by providing the strict convergence proof for nonlocal problems.

MGMs are among the most efficient iterative methods for solving large-scale systems of equations, arising from the discretization of PDEs [10, 25, 46]. Using the regularity and finite element approximation theory, the convergence estimates of the V-cycle MGM have been proved for elliptic PDEs [6, 10, 11] and fractional differential equations (FDEs) [15, 16, 28]. By using the compact notion of the symbol and its basic analytical features, the V-cycle optimal convergence has been derived for the case of multilevel linear systems whose coefficient matrices belong to the circulant, Hartely, or τ algebras or to the Toeplitz class [4, 5, 9, 39]. It directly tackles the stiffness matrix of the resulting algebraic system, which can often be derived directly from the underlying matrices, without any reference to the grids. In general, when considering the discretization of PDEs/FDEs, it is hard to make a theoretically concise statement [45, 46], but it is possible to give a rather simple analysis [45] for a very special one-dimensional elliptic PDE and the two-dimensional case [19]. However, it is still not at all easy for the dense stiffness matrices [4, 5, 9, 18], unless we can reduce the problem to the Toeplitz setting and we know the symbol, its zeros, and their orders [39]. Instead, we will focus our attention on first answering such a question for a two-level setting since it is useful from a theoretical point of view as the first step to study the MGM convergence, usually beginning with the two-grid method (TGM) [33, 35, 36, 46].

As is well known, a theoretical analysis of the related TGM is given in terms of the algebraic multigrid theory considered in [35]. For solving Toeplitz systems, the convergence of the TGM on the first level was proved for the so-called band τ matrices in [24], and a complete analysis of convergence of the TGM was given for elliptic Toeplitz and PDEs matrix-sequences [39]. For a class of weakly diagonally dominant Toeplitz matrices, the uniform convergence of the TGM was theoretically obtained in [13] and extended to nonlocal operators in [17, 20, 33]. For Toeplitz-plus-diagonal systems, the numerical behavior of MGM have been discussed in [32] (see also [40] for the TGM convergence), and the preconditioned Krylov subspace methods, including the conjugate gradient method, have been proposed in [22, 31, 34, 44]. It should be noted that the proof technique is different, and in fact the related proofs do not rely on the diagonal dominance [4, 5, 13, 39]. In reality, the two-grid optimality (and sometimes the V-cycle optimality) is proven for Toeplitz matrices $T_n(f)$, where f is nonnegative and has isolated zeros: The proof and the algorithm depend on the position and order of such zeros, and most of such matrices are far from being diagonally dominant [4, 5, 39]. For Toeplitz-plus-tridiagonal systems arising from nonlocal problems, to the best of our knowledge, no fast MGMs have been developed, and no convergence analysis has been provided; moreover, there are still questions for the MGM when the system matrix is far from being weakly diagonally dominant [42]. In this paper, first we give a structural and spectral analysis of the underlying matrices. Then, based on the latter study, the simple (traditional) restriction operator and prolongation operator are employed in order to handle general algebraic systems: Here the stiffness matrices of the resulting systems are Toeplitz plus tridiagonal and far from being diagonally dominant, respectively, corresponding to the fractional Laplacian kernel and the constant kernel. We focus on providing the detailed proof of the convergence of the TGM for such situations. Moreover, the convergence of the full multigrid, i.e., recursive application of the TGM procedure [13, 24, 39], is also discussed with the constant kernel. The performed numerical experiments show the effectiveness of the MGM with only $\mathcal{O}(N\log N)$ complexity by the fast Fourier transform, where N is the number of grid points.

The outline of this paper is as follows. In the next section, we derive the algebraic systems for the nonlocal problems arising from linear finite element approximations, we study their structural and spectral features, and we introduce the MGM algorithms. In section 3, we study the uniform convergence estimates of the TGM for the considered nonlocal problem with the fractional Laplacian kernel. Convergence of the TGM and full MGM with a constant kernel case is analyzed in section 4. To show the effectiveness of the presented schemes, results of numerical experiments are reported in section 5. Finally, we conclude the paper with remarks on the presented results and by stating a few open questions.

2. Preliminaries: Numerical scheme.

In this section, we derive the numerical discretization for nonlocal diffusion problems with the fractional Laplacian kernel and the constant kernel, respectively. Let Ω be a finite bar in $\mathbb{R}$. The nonlocal operator is used in the time-dependent nonlocal diffusion problem [3, 7, 30]

$$\begin{cases} u_t(x,t) - \mathcal{L}u(x,t) = f(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \mathbb{R} \setminus \Omega, \end{cases}$$

and its steady-state counterpart

$$\begin{cases} -\mathcal{L}u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega. \end{cases} \tag{2.1}$$

Here, the nonlocal operator $\mathcal{L}$ is defined by

$$\mathcal{L}u(x) = \int_\Omega [u(y) - u(x)] J(|x-y|) dy \quad \forall x \in \Omega,$$

where J is a radial probability density with a nonnegative symmetric dispersal kernel.

From [30], we obtain

$$\begin{aligned} (-\mathcal{L}u, v) &= \int_\Omega v(x) \int_\Omega [u(x) - u(y)] J(|x-y|) dy dx \\ &= \frac{1}{2} \int_\Omega \int_\Omega [u(y) - u(x)][v(y) - v(x)] J(|x-y|) dy dx. \end{aligned}$$

The energy space associated with (2.1) is defined as

$$\mathcal{S}(\Omega) = \{u \in L^2(\Omega) \big| \int_\Omega \int_\Omega [u(y) - u(x)]^2 J(|x-y|) dy dx < \infty\},$$

and $\mathcal{S}_0(\Omega) = \{u \in \mathcal{S}(\Omega) \big| u = 0 \text{ in } \mathbb{R} \setminus \Omega\}$.

Formally, we can define a bilinear form $a(u,v) : \mathcal{S}_0(\Omega) \times \mathcal{S}_0(\Omega) \to \mathbb{R}$ by

$$a(u,v) = \frac{1}{2} \int_\Omega \int_\Omega [u(y) - u(x)][v(y) - v(x)] J(|x-y|) dy dx,$$

and the weak formulation of (2.1) is expressed as follows: Find $u \in \mathcal{S}_0(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in \mathcal{S}_0(\Omega). \tag{2.2}$$

It was proved that the bilinear form $a(u,v)$ is coercive and bounded on the nonconventional Hilbert space $\mathcal{S}_0(\Omega)$, and second-order convergence can be expected for the linear finite element, with sufficiently smooth functions [23, 30]. Let $\Omega = (0,b)$ with the mesh points $x_i = ih$, $h = b/N$ and u_i as the numerical approximation of $u(x_i)$, and $f_i = f(x_i)$. Denote $I_i = ((i-1)h, ih)$ for $1 \le i \le N$, and the piecewise linear basis function is

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}, \end{cases}$$

with $i = 1, 2, \ldots, N-1$, and

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h}, & x \in [x_0, x_1], \\ 0, & \text{otherwise}, \end{cases} \qquad \phi_N(x) = \begin{cases} \dfrac{x - x_{N-1}}{h}, & x \in [x_{N-1}, b], \\ 0, & \text{otherwise}. \end{cases}$$

Let $\mathcal{S}_0^h(\Omega) \subset \mathcal{S}_0(\Omega)$ be the finite element space consisting of piecewise linear polynomials $\phi_i(x)$ with respect to the uniform mesh. The finite element approximation of the variational problem (2.2) is expressed in accordance with the continuous setting: Find $u_h \in \mathcal{S}_0^h(\Omega)$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in \mathcal{S}_0^h(\Omega). \tag{2.3}$$

Using $u_h = \sum_{i=1}^{N-1} u_i \phi_i(x)$, we can rewrite (2.3) as

$$\sum_{i=1}^{N-1} a(\phi_i, \phi_j) u_i = (f, \phi_j), \quad j = 1, 2, \ldots, N-1, \tag{2.4}$$

which is representable as a linear system of equations

$$A_h u_h = f_h, \quad (A_h)_{i,j} = a_{i,j} = a(\phi_i, \phi_j), \quad \text{and} \quad (f_h)_j = (f, \phi_j). \tag{2.5}$$

Here, $u_h = [u_1, u_2, \ldots, u_{N-1}]^{\mathrm{T}}$, and the matrix A_h is known as the stiffness matrix of the nodal basis $\{\phi_i\}_{i=1}^{N-1}$. More concretely,

$$(A_h)_{i,j} = a_{i,j} = a(\phi_i, \phi_j) = \int_{x_{i-1}}^{x_{i+1}} \phi_i(x) \left[\int_\Omega [\phi_j(x) - \phi_j(y)] J(|x-y|) dy \right] dx. \tag{2.6}$$

In this paper, we mainly focus on two types of the classical kernel functions for (2.1), i.e., the fractional Laplacian kernel [3, 23] and the constant kernel [2, 43]. More general kernel types [2, 3, 23, 43] can be similarly studied.

2.1. Discretization scheme for (2.1) with fractional Laplacian kernel: A Toeplitz-plus-tridiagonal system. In this subsection, we choose the fractional Laplacian kernel $J(|x|) = C_\alpha / |x|^{1+\alpha}$ with $\alpha \in (1,2)$; then (2.1) reduces to the integral version fractional Laplacian

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) := C_\alpha \displaystyle\int_\Omega \frac{u(x) - u(y)}{|x-y|^{1+\alpha}} dy = f(x), & x \in \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad u(x) = 0, & x \in \mathbb{R} \setminus \Omega, \end{cases} \tag{2.7}$$

with

$$C_\alpha = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{1+\alpha}{2})}{\pi^{1/2}\Gamma(1-\alpha/2)} = \kappa_\alpha \frac{-\alpha}{\Gamma(1-\alpha)} > 0, \quad \kappa_\alpha = \frac{-1}{2\cos(\alpha\pi/2)} > 0, \quad \alpha \in (1,2).$$

It should be noted that there exist several equivalent definitions of the fractional Laplacian, agreeing with the space of appropriately smooth functions [29].

From (2.6) and (2.7), the entries $a_{i,j}$ $(j \geq i+2)$ of the matrix are given by

$$\begin{aligned}
a_{i,j} &= a(\phi_i, \phi_j) \\
&= C_\alpha \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h} \left[\int_{x_{j-1}}^{x_j} \frac{0 - \frac{y - x_{j-1}}{h}}{(y-x)^{1+\alpha}} dy + \int_{x_j}^{x_{j+1}} \frac{0 - \frac{x_{j+1}-y}{h}}{(y-x)^{1+\alpha}} dy \right] dx \\
&\quad + C_\alpha \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{h} \left[\int_{x_{j-1}}^{x_j} \frac{0 - \frac{y - x_{j-1}}{h}}{(y-x)^{1+\alpha}} dy + \int_{x_j}^{x_{j+1}} \frac{0 - \frac{x_{j+1}-y}{h}}{(y-x)^{1+\alpha}} dy \right] dx \\
&= -C_\alpha h^{1-\alpha} \left[\int_0^1 t \int_0^1 \frac{s}{(j-i+s-t)^{1+\alpha}} ds dt + \int_0^1 t \int_0^1 \frac{s}{(j-i+2-s-t)^{1+\alpha}} ds dt \right. \\
&\quad \left. + \int_0^1 t \int_0^1 \frac{s}{(j-i-2+s+t)^{1+\alpha}} ds dt + \int_0^1 t \int_0^1 \frac{s}{(j-i-s+t)^{1+\alpha}} ds dt \right].
\end{aligned}$$

Similarly, we can obtain $a_{i,i}$ and $a_{i,i+1} = a_{i+1,i}$. By calculation, it is easy to get

$$A_h = \frac{\kappa_\alpha}{h^{\alpha-1}\Gamma(4-\alpha)} B_h, \tag{2.8}$$

where the entries of the stiffness matrix $(B_h)_{i,j} = b_{i,j}$ are explicitly given by

$$\begin{aligned}
b_{i,i} &= \left(8 - 2^{4-\alpha}\right) + \widetilde{b}_{i,i}, \ i = 1, 2, \ldots, N-1, \\
b_{i,i+1} &= b_{i+1,i} = \left(-7 - 3^{3-\alpha} + 2^{5-\alpha}\right) + \widetilde{b}_{i,i+1}, \ i = 1, 2, \ldots, N-1, \\
b_{i,j} &= -\left(m+2\right)^{3-\alpha} + 4\left(m+1\right)^{3-\alpha} \\
&\qquad - 6m^{3-\alpha} + 4\left(m-1\right)^{3-\alpha} - \left(m-2\right)^{3-\alpha}, \ m = |j-i| \geq 2,
\end{aligned} \tag{2.9}$$

with

$$\begin{aligned}
\widetilde{b}_{i,i} &= 2\left[(i+1)^{3-\alpha} - 2(3-\alpha)i^{2-\alpha} - (i-1)^{3-\alpha}\right] \\
&\quad + 2\left[(N-i+1)^{3-\alpha} - 2(3-\alpha)(N-i)^{2-\alpha} - (N-i-1)^{3-\alpha}\right], \\
\widetilde{b}_{i,i+1} &= -2\left[(i+1)^{3-\alpha} - i^{3-\alpha}\right] + (3-\alpha)\left[(i+1)^{2-\alpha} + i^{2-\alpha}\right] \\
&\quad - 2\left[(N-i)^{3-\alpha} - (N-i-1)^{3-\alpha}\right] + (3-\alpha)\left[(N-i)^{2-\alpha} + (N-i-1)^{2-\alpha}\right].
\end{aligned}$$

In fact, the stiffness matrix A_h (or equivalently B_h) is a symmetric diagonally dominant Toeplitz-plus-tridiagonal matrix (which we will prove in Lemma 4.3), i.e.,

$$B_h = T_h + E_h.$$

Here,

$$T_h = \begin{bmatrix} c_0 & c_1 & \cdots & c_{N-2} \\ c_1 & c_0 & \cdots & c_{N-3} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N-2} & c_{N-3} & \cdots & c_0 \end{bmatrix}_{(N-1)\times(N-1)},$$

with

$$c_m = \begin{cases} 8 - 2^{4-\alpha}, & m = 0, \\ -7 - 3^{3-\alpha} + 2^{5-\alpha}, & m = 1, \\ -(m+2)^{3-\alpha} + 4(m+1)^{3-\alpha} - 6m^{3-\alpha} + 4(m-1)^{3-\alpha} - (m-2)^{3-\alpha}, & m \geq 2, \end{cases}$$

and

$$E_h = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{N-3} & a_{N-2} & b_{N-2} \\ & & & b_{N-2} & a_{N-1} \end{bmatrix}_{(N-1)\times(N-1)},$$

where

$$a_i = \widetilde{b}_{i,i}, \quad 1 \leq i \leq N-1,$$
$$b_i = \widetilde{b}_{i,i+1}, \quad 1 \leq i \leq N-2.$$

2.2. Discretization scheme for (2.1) with constant kernel: A non–diagonally dominant system. For simplicity, we choose the constant kernel $J(|x|) = 1$; then (2.1) reduces to the following pseudodifferential equation:

$$\begin{cases} \displaystyle\int_\Omega [u(x) - u(y)]\, dy = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega. \end{cases} \tag{2.10}$$

Using (2.6) and (2.10), it is immediate to obtain

$$A_h = h^2 B_h, \tag{2.11}$$

with

$$B_h = \begin{bmatrix} \frac{2N}{3}-1 & \frac{N}{6}-1 & -1 & \cdots & -1 \\ \frac{N}{6}-1 & \frac{2N}{3}-1 & \ddots & \ddots & \vdots \\ -1 & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \frac{2N}{3}-1 & \frac{N}{6}-1 \\ -1 & \cdots & -1 & \frac{N}{6}-1 & \frac{2N}{3}-1 \end{bmatrix}_{(N-1)\times(N-1)};$$

i.e., the entries of the stiffness matrix $(B_h)_{i,j} = b_{i,j}$ are explicitly given by

$$b_{i,j} = b_{|i-j|} = b_k = \begin{cases} 2N/3 - 1, & k = 0, \\ N/6 - 1, & k = 1, \\ -1, & \text{otherwise}. \end{cases} \tag{2.12}$$

2.3. Spectral analysis of the scaled matrices $B'_h = B_h/N$ in (2.11). A matrix of size n, having a fixed entry along each diagonal, is called Toeplitz. Given a complex-valued Lebesgue integrable function $\phi : [-\pi, \pi] \to \mathbb{C}$, the nth Toeplitz matrix generated by ϕ is defined as [14]

$$T_n(\phi) = \left[\hat{\phi}_{i-j}\right]_{i,j=1}^{n},$$

where the quantities $\hat{\phi}_k$ are the Fourier coefficients of ϕ, that is,

$$\hat{\phi}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\theta)\,\mathrm{e}^{-\mathbf{i}k\theta}\mathrm{d}\theta, \qquad k \in \mathbb{Z}.$$

We refer to $\{T_n(\phi)\}_n$ as the Toeplitz sequence generated by ϕ, which in turn is called the generating function of $\{T_n(\phi)\}_n$. In the case where ϕ is real-valued, all the matrices $T_n(\phi)$ are Hermitian, and much is known about their spectral properties.

More in detail, if ϕ is real-valued and not identically constant, then any eigenvalue of $T_n(\phi)$ belongs to the open set (m_ϕ, M_ϕ), with m_ϕ and M_ϕ being the essential infimum and the essential supremum of ϕ, respectively; see [38]. The case of a constant ϕ is trivial: in that case, if $\phi = m$ almost everywhere, then $T_n(\phi) = mI_n$ with I_n denoting the identity of size n. Hence, if $M_\phi > 0$ and ϕ is nonnegative almost everywhere, then $T_n(\phi)$ is Hermitian positive definite.

Now we consider $B'_h = B_h/N$, B_h being the matrix defined in (2.11). From the coefficients in (2.12) and taking into account the definition of the generating function above, we have

$$B'_h = T_{N-1}(g(\theta)) - \frac{1}{N}ee^{\mathrm{T}}, \qquad g(\theta) = \frac{2}{3} + \frac{1}{3}\cos(\theta), \tag{2.13}$$

with $e = [1, 1, \ldots, 1]^{\mathrm{T}}$ being the vector of all ones of size $N-1$. By using the analysis in [38], we know that the eigenvalues of $T_{N-1}(g(\theta))$ belong to the open set $(\frac{1}{3}, 1)$ since $\frac{1}{3} = \min g(\theta)$, $1 = \max g(\theta)$. Furthermore, by ordering the eigenvalues nonincreasingly, since $T_{N-1}(g(\theta))$ belongs to a sine-transform algebra (the so-called τ algebra [37]), we have

$$\lambda_j(T_{N-1}(g(\theta))) = g\left(\frac{j\pi}{N}\right), \qquad j = 1, \ldots, N-1. \tag{2.14}$$

Here the rank-one correction $-\frac{1}{N}ee^{\mathrm{T}}$ is nonpositive definite with the unique nonzero eigenvalue equal to the trace, that is, $-\frac{N-1}{N}$. Therefore, the use of the Cauchy interlacing results implies

$$\frac{1}{3} < \lambda_{j+1}(T_{N-1}(g(\theta))) = g\left(\frac{(j+1)\pi}{N}\right) \le \lambda_j(B'_h) \le \lambda_j(T_{N-1}(g(\theta))) = g\left(\frac{j\pi}{N}\right) < 1 \tag{2.15}$$

for $j = 1, \ldots, N-2$, and

$$\frac{1}{3} - \frac{N-1}{N} \le \lambda_{N-1}(B'_h) \le \lambda_{N-1}(T_{N-1}(g(\theta))) = g\left(\frac{(N-1)\pi}{N}\right) \approx \frac{1}{3}. \tag{2.16}$$

While the estimates in (2.15) are tight, the last estimate in (2.16) for the minimal eigenvalue is poor. We can improve it by exploiting the fact that the matrix B_h is obtained from a Galerkin approximation of a coercive operator, and therefore

$$\lambda_{N-1}(B'_h) \in \left(0, \lambda_{N-1}(T_{N-1}(g(\theta)))\right), \qquad \lambda_{N-1}(T_{N-1}(g(\theta))) = g\left(\frac{(N-1)\pi}{N}\right) \approx \frac{1}{3}.$$

Still the localization is not precise, and in the following, we employ more advanced tools for evaluating the asymptotic behavior of the minimal eigenvalue and of the spectral conditioning of B_h' (and hence of B_h).

First of all we exploit the low-frequency vector e in connection with the Rayleigh quotient for Hermitian matrices. We have

$$\begin{aligned}\lambda_{N-1}(B_h') = \lambda_{\min}(B_h') &= \min_{x\neq 0} \frac{x^{\mathrm{T}} B_h' x}{x^{\mathrm{T}} x} \\ &\leq \frac{e^{\mathrm{T}} B_h' e}{e^{\mathrm{T}} e} = \frac{e^{\mathrm{T}} T_{N-1}(g(\theta)) e - \frac{1}{N}[e^{\mathrm{T}} e]^2}{e^{\mathrm{T}} e} \\ &= \frac{N-1-\frac{1}{3}-\frac{(N-1)^2}{N}}{N-1} = \frac{2}{3N} + O(N^{-2}),\end{aligned}$$

and therefore

$$\lambda_{N-1}(B_h') \in \left(0, \frac{2}{3N} + O(N^{-2})\right), \tag{2.17}$$

which implies that the coefficient matrix B_h is asymptotically ill-conditioned and its conditioning grows at least as $3N/2$.

In the following, using special properties of rank-one matrices, we show that the conditioning is exactly growing proportionally to N, which implies that the vector e is a good approximation of the related eigenvector. We set $X = T_{N-1}(g(\theta))$, and given its invertibility, we can write

$$B_h' = X\left[I_{N-1} - \frac{1}{N} X^{-1} e e^{\mathrm{T}}\right].$$

Now the matrix $-\frac{1}{N} X^{-1} e e^{\mathrm{T}}$ is still a rank-one matrix, and its unique nonzero eigenvalue coincides with its trace, that is,

$$-\frac{1}{N} e^{\mathrm{T}} X^{-1} e.$$

From the latter, we deduce

$$\det(B_h') = \det(X)\det\left(I_{N-1} - \frac{1}{N} X^{-1} e e^{\mathrm{T}}\right) = \det(X)\left(1 - \frac{1}{N} e^{\mathrm{T}} X^{-1} e\right). \tag{2.18}$$

Finally, since

$$\lambda_{N-1}(B_h') = \lambda_{\min}(B_h') = \frac{\det(B_h')}{\lambda_1(B_h')\cdots\lambda_{N-2}(B_h')},$$

from (2.15), we obtain

$$\lambda_{\min}(X)\left(1 - \frac{1}{N} e^{\mathrm{T}} X^{-1} e\right) \leq \lambda_{\min}(B_h') \leq \lambda_{\max}(X)\left(1 - \frac{1}{N} e^{\mathrm{T}} X^{-1} e\right),$$

with

$$\lambda_{\min}(X) = g\left(\frac{(N-1)\pi}{N}\right) = \frac{1}{3} + \mu(N), \quad \lambda_{\max}(X) = g\left(\frac{\pi}{N}\right) = 1 - \nu(N),$$

with $\mu(N), \nu(N) \sim N^{-2}$.

In other words, up to a quantity in the interval $(\frac{1}{3}, 1)$, the minimal eigenvalue of B'_h is asymptotic to $1 - \frac{1}{N} e^{\mathrm{T}} X^{-1} e$, where a detailed but cumbersome analysis leads to

$$\left(1 - \frac{1}{N} e^{\mathrm{T}} X^{-1} e\right) N\sqrt{3} = 1 + O(N^{-1}),$$

so that

$$\lambda_{N-1}(B'_h) \sim 1 - \frac{1}{N} e^{\mathrm{T}} X^{-1} e \approx \frac{1}{\sqrt{3}N} + O(N^{-2}). \tag{2.19}$$

We now comment on the results obtained in this subsection.

(A) From the relation (2.13), we observe that neither B_h nor B'_h can be written as a Toeplitz matrix generated by a symbol independent of N; hence, the proof techniques developed for the TGM/MGM optimality developed in [5, 39] cannot be applied directly. In section 5, we proceed with an alternative approach.

(B) Again relation (2.13) implies that the matrix B'_h is a rank-one correction of the well-conditioned Toeplitz matrix $T_{N-1}(g(\theta))$ (with spectral conditioning strictly bounded by 3). Therefore, a corresponding linear system can be solved with linear complexity by using the Sherman–Morrison–Woodbury formula or a preconditioned conjugate gradient with preconditioner given exactly by $T_{N-1}(g(\theta))$. However, the proof technique used in this setting is also adaptable to the case of a general α with the tools in [40], while, changing α, both the rank structure and the Toeplitz structure are not preserved.

(C) The analysis presented in the previous items tells us that the matrix B'_h is asymptotically ill-conditioned and that the responsibility for the ill-conditioning is the vector e, which is a special instance of a low-frequency vector. We recall that in the discretization of elliptic equations, the low-frequency subspace is associated with the small eigenvalues. Hence, the latter observation suggests that an appropriate restriction operator and prolongation operators can be chosen as the traditional ones for elliptic problems. This choice will be employed in the next sections.

3. MGM.

Consider an algebraic system $A_h u_h = f_h$, where $u_h \in \mathcal{R}^{n_q}$ and n_q is the size of the matrix A_h. We define a sequence of subsystems on different levels:

$$A_m u_m = f_m,\ u_m \in \mathcal{R}^{n_m}, \quad m = 1, \ldots, q.$$

Here q is the total number of levels, with $m = q$ being the finest level, i.e., $A_q = A_h$. For $m \geq 1$, n_m are just the size of the matrix A_m.

The traditional (simple) restriction operator I_m^{m-1} and prolongation operator I_{m-1}^m are, respectively, defined by

$$\nu^{m-1} = I_m^{m-1}\nu^m \ \text{ with } \ \nu_i^{m-1} = \frac{1}{4}\left(\nu_{2i-1}^m + 2\nu_{2i}^m + \nu_{2i+1}^m\right), \quad i = 1, \ldots, \mathcal{R}^{n_{m-1}},$$

and

$$\nu^m = I_{m-1}^m \nu^{m-1} \ \text{ with } \ I_{m-1}^m = 2\left(I_m^{m-1}\right)^{\mathrm{T}}.$$

We use the coarse grid operators defined by the Galerkin approach [36, p. 455]

$$A_{m-1} = I_m^{m-1} A_m I_{m-1}^m, \tag{3.1}$$

and for all the intermediate $(m, m-1)$ coarse grids, we apply the correction operators [35, p. 87]

$$T^m = I_m - I_{m-1}^m A_{m-1}^{-1} I_m^{m-1} A_m = I_m - I_{m-1}^m P_{m-1},$$

with

$$P_{m-1} = A_{m-1}^{-1} I_m^{m-1} A_m.$$

We choose the damped Jacobi iteration matrix by [12, p. 9]

$$K_m = I - S_m A_m \quad \text{with} \quad S_m := S_{m,\omega} = \omega D_m^{-1}, \tag{3.2}$$

with a weighting factor ω, and D_m is the diagonal of A_m.

A multigrid process can be regarded as defining a sequence of operators $B_m : \mathcal{R}^{n_m} \mapsto \mathcal{R}^{n_m}$ which is an approximate inverse of A_m in the sense that $\|I - B_m A_m\|$ is bounded away from one. The V-cycle multigrid algorithm [45] is provided in Algorithm 1. If $m = 2$, the resulting Algorithm 1 is TGM.

Algorithm 1 V-cycle Multigrid Algorithm: Define $B_1 = A_1^{-1}$. Assume that $B_{m-1} : \mathcal{R}^{n_{m-1}} \mapsto \mathcal{R}^{n_{m-1}}$ is defined. We shall now define $B_m : \mathcal{R}^{n_m} \mapsto \mathcal{R}^{n_m}$ as an approximate iterative solver for the equation associated with $A_m \nu_m = f_m$.

1: Presmooth: Let $S_{m,\omega}$ be defined by (3.2) and $\nu_m^0 = 0$, $l = 1, \ldots, m_1$:

$$\nu_m^l = \nu_m^{l-1} + S_{m,\omega_{pre}}(f_m - A_m \nu_m^{l-1}).$$

2: Coarse grid correction: $e^{m-1} \in \mathcal{R}^{n_{m-1}}$ is the approximate solution of the residual equation $A_{m-1} e = I_m^{m-1}(f_m - A_m \nu_m^{m_1})$ by the iterator B_{m-1}:

$$e^{m-1} = B_{m-1} I_m^{m-1}(f_m - A_m \nu_m^{m_1}).$$

3: Postsmooth: $\nu_m^{m_1+1} = \nu_m^{m_1} + I_{m-1}^m e^{m-1}$ and $l = m_1 + 2, \ldots, m_1 + m_2$

$$\nu_m^l = \nu_m^{l-1} + S_{m,\omega_{post}}(f_m - A_m \nu_m^{l-1}).$$

4: Define $B_m f_m = \nu_m^{m_1+m_2}$.

3.1. A few technical lemmas. First, we give some lemmas that will be used later on.

LEMMA 3.1 ([19]). *Let $A^{(1)} = \{a_{i,j}^{(1)}\}_{i,j=1}^\infty$ with $a_{i,j}^{(1)} = a_{|i-j|}^{(1)}$ be a symmetric Toeplitz matrix and $A^{(k)} = L_h^H A^{(k-1)} L_H^h$ with $L_h^H = 4 I_k^{k-1}$ and $L_H^h = (L_h^H)^{\mathrm{T}}$. Then $A^{(k)}$ can be computed by*

$$a_0^{(k)} = (4C_k + 2^{k-1}) a_0^{(1)} + \sum_{m=1}^{2\cdot 2^{k-1}-1} {}_0C_m^k a_m^{(1)},$$

$$a_1^{(k)} = C_k a_0^{(1)} + \sum_{m=1}^{3\cdot 2^{k-1}-1} {}_1C_m^k a_m^{(1)},$$

$$a_j^{(k)} = \sum_{m=(j-2)2^{k-1}}^{(j+2)2^{k-1}-1} {}_jC_m^k a_m^{(1)} \quad \forall j, k \geq 2,$$

with $C_k = 2^{k-2} \cdot \frac{2^{2k-2}-1}{3}$. *Furthermore,*

$$
{}_0C_m^k = \begin{cases} 8C_k - (m^2-1)(2^k - m), & m = 1, \dots, 2^{k-1}, \\ \frac{1}{3}(2^k - m - 1)(2^k - m)(2^k - m + 1), & m = 2^{k-1}, \dots, 2 \cdot 2^{k-1} - 1, \end{cases}
$$

$$
{}_1C_m^k =
$$

$$
\begin{cases} 2C_k + m^2 \cdot 2^{k-1} - \dfrac{2}{3}(m-1)m(m+1), & m = 1, \dots, 2^{k-1}, \\ 2C_k + (2^k - m)^2 \cdot 2^{k-1} - \dfrac{2}{3}(2^k - m - 1)(2^k - m)(2^k - m + 1) & \\ \quad - \dfrac{1}{6}(m - 2^{k-1} - 1)(m - 2^{k-1})(m - 2^{k-1} + 1), & m = 2^{k-1}, \dots, 2 \cdot 2^{k-1}, \\ \dfrac{1}{6}(3 \cdot 2^{k-1} - m - 1)(3 \cdot 2^{k-1} - m)(3 \cdot 2^{k-1} - m + 1), & m = 2 \cdot 2^{k-1}, \dots, 3 \cdot 2^{k-1} - 1, \end{cases}
$$

and for $j \geq 2$,

$$
{}_jC_m^k = \begin{cases} \varphi_1, & m = (j-2)2^{k-1}, \dots, (j-1)2^{k-1}, \\ \varphi_2, & m = (j-1)2^{k-1}, \dots, j2^{k-1}, \\ \varphi_3, & m = j2^{k-1}, \dots, (j+1)2^{k-1}, \\ \varphi_4, & m = (j+1)2^{k-1}, \dots, (j+2)2^{k-1} - 1, \end{cases}
$$

where

$$
\begin{aligned}
\varphi_1 &= \frac{1}{6}(m - (j-2)2^{k-1} - 1)(m - (j-2)2^{k-1})(m - (j-2)2^{k-1} + 1), \\
\varphi_2 &= 2C_k + (m - (j-1)2^{k-1})^2 \cdot 2^{k-1} \\
&\quad - \frac{1}{6}(j2^{k-1} - m - 1)(j2^{k-1} - m)(j2^{k-1} - m + 1) \\
&\quad - \frac{2}{3}(m - (j-1)2^{k-1} - 1)(m - (j-1)2^{k-1})(m - (j-1)2^{k-1} + 1), \\
\varphi_3 &= 2C_k + ((j+1)2^{k-1} - m)^2 \cdot 2^{k-1} \\
&\quad - \frac{1}{6}(m - j2^{k-1} - 1)(m - j2^{k-1})(m - j2^{k-1} + 1) \\
&\quad - \frac{2}{3}((j+1)2^{k-1} - m - 1)((j+1)2^{k-1} - m)((j+1)2^{k-1} - m + 1), \\
\varphi_4 &= \frac{1}{6}((j+2)2^{k-1} - m - 1)((j+2)2^{k-1} - m)((j+2)2^{k-1} - m + 1).
\end{aligned}
$$

LEMMA 3.2. *Let* $E^{(1)}$ *be a symmetric tridiagonal matrix and* $E^{(k)} = L_h^H E^{(k-1)} L_H^h$ *with* $L_h^H = 4I_k^{k-1}$ *and* $L_H^h = (L_h^H)^{\mathrm{T}}$. *Then* $E^{(k)}$ *is a symmetric tridiagonal matrix for all* k.

Proof. Let q be a total number of levels with $N = 2^q$ and

$$
E^{(1)} = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{N-3} & a_{N-2} & b_{N-2} \\ & & & b_{N-2} & a_{N-1} \end{bmatrix}_{(N-1)\times(N-1)}.
$$

Then we have

$$E^{(2)} = \begin{bmatrix} d_1 & e_1 & & & \\ e_1 & d_2 & e_2 & & \\ & \ddots & \ddots & \ddots & \\ & & e_{N/2-3} & d_{N/2-2} & e_{N/2-2} \\ & & & e_{N/2-2} & d_{N/2-1} \end{bmatrix}_{(N/2-1)\times(N/2-1)},$$

with

$$\begin{aligned} d_i &= a_{2i-1} + 4(b_{2i-1} + b_{2i} + a_{2i}) + a_{2i+1}, \quad 1 \le i \le N/2-1, \\ e_i &= a_{2i+1} + 2(b_{2i} + b_{2i+1}), \quad 1 \le i \le N/2-2. \end{aligned}$$

By mathematical induction, the proof is completed. ∎

3.2. The operation count and storage requirement. We now discuss the computation count and the required storage for the MGM of the nonlocal problems (2.1).

From (2.9), we know that the matrix A_h is a symmetric Toeplitz-plus-tridiagonal matrix. Then we only need to store the first column, principal diagonal, and trailing diagonal elements of A_h, which have $\mathcal{O}(N)$ parameters, instead of the full matrix A_h with N^2 entries. From Lemmas 3.1 and 3.2, we know that $\{A_k\}$ is still a symmetric Toeplitz-plus-tridiagonal matrix with the sizes $2^{k-q}\mathcal{O}(N)$ storage. Adding these terms together, we have

$$\text{Storage} = \mathcal{O}(N) \cdot \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{q-1}}\right) = \mathcal{O}(N).$$

As for operation counts, the matrix-vector product associated with the matrix A_h is a discrete convolution. While the cost of a direct product is $O(N)$ for tridiagonal matrix, the cost of using the fast Fourier transform would lead to $O(N\log(N)$ for dense Toeplitz matrix [14]. Thus, the total per V-cycle MGM operation count is

$$\text{Operation count} = \mathcal{O}(N\log N) \cdot \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{q-1}}\right) = \mathcal{O}(N\log N).$$

Similarly, we can discuss the case of the matrix A_h in (2.11).

4. Convergence of TGM for (2.7): A Toeplitz-plus-tridiagonal system. Now, we start to prove the convergence of the TGM for nonlocal problem (2.7) with the fractional Laplacian kernel, which is a special Toeplitz-plus-tridiagonal system. First, we give some lemmas that will be used. Since the matrix A_h is symmetric positive definite, we can define the following inner products:

$$(u,v)_D = (Du, v), \quad (u,v)_A = (Au, v), \quad \text{and} \quad (u,v)_{AD^{-1}A} = (Au, Av)_{D^{-1}},$$

where $A := A_q = A_h$, D is its diagonal, and $(\cdot,\cdot)$ is the usual Euclidean inner product.

Lemma 4.1 ([35, p. 84]). *Let A_m be a symmetric positive definite matrix. If $\eta \le \omega(2-\omega\eta_0)$ with $\eta_0 \ge \lambda_{\max}(D_m^{-1}A_m)$, then the damped Jacobi iteration with relaxation parameter $0 < \omega < 2/\eta_0$ satisfies*

$$\|K_m\nu^m\|_{A_m}^2 \le \|\nu^m\|_{A_m}^2 - \eta\|A_m\nu^m\|_{D_m^{-1}}^2 \quad \forall \nu^m \in \mathcal{R}^{n_m}, \quad m = 1, \ldots, q. \tag{4.1}$$

LEMMA 4.2 ([13, 24, 35]). *Let A_m be a symmetric positive definite matrix and K_m satisfy (4.1) and*

$$\min_{\nu^{m-1}\in\mathcal{R}^{n_{m-1}}} \|\nu^m - I_{m-1}^m \nu^{m-1}\|_{D_m}^2 \le \kappa \|\nu^m\|_{A_m}^2 \quad \forall \nu^m \in \mathcal{R}^{n_m}, \quad m = 1, \ldots, q, \tag{4.2}$$

with $\kappa > 0$ independent of ν^m. Then $\kappa \ge \eta > 0$, and the convergence factor of full MGM satisfies

$$\|K_m T^m\|_{A_m} \le \sqrt{1-\eta/\kappa} \quad \forall \nu^m \in \mathcal{R}^{n_m}, \quad m = 1, \ldots, q.$$

In particular, the convergence factor of TGM satisfies

$$\|K_q T^q\|_{A_q} \le \sqrt{1-\eta/\kappa} \quad \forall \nu^q \in \mathcal{R}^{n_q}.$$

LEMMA 4.3. *Let $1 < \alpha < 2$. The coefficients $b_{i,j}$ with $1 \le i, j \le N-1$ in (2.9) satisfy*

$$\begin{aligned} &(1) \quad b_{i,j} < 0, \ j \ne i \quad \text{and} \quad \widetilde{b}_{i,i+1} < 0, \ \widetilde{b}_{i,i} < 0, \\ &(2) \quad \sum_{j=1}^{N-1} b_{i,j} > 0 \quad \text{and} \quad b_{i,i} > \sum_{j \ne i} |b_{i,j}| > 0. \end{aligned}$$

Proof. (I) We first prove $b_{i,j} < 0$ with $m = |j-i| \ge 2$. From (2.9), we have

$$\begin{aligned} b_{i,j} &= -m^{3-\alpha}\left[\left(1+\frac{2}{m}\right)^{3-\alpha} - 4\left(1+\frac{1}{m}\right)^{3-\alpha} + 6 - 4\left(1-\frac{1}{m}\right)^{3-\alpha} + \left(1-\frac{2}{m}\right)^{3-\alpha}\right] \\ &= -m^{3-\alpha}\left\{\sum_{k=4}^{\infty}\binom{3-\alpha}{k}\left[\left(\frac{2}{m}\right)^k - 4\left(\frac{1}{m}\right)^k - 4\left(\frac{-1}{m}\right)^k + \left(\frac{-2}{m}\right)^k\right]\right\} \\ &= -m^{3-\alpha}\left\{\sum_{k=4}^{\infty}\binom{3-\alpha}{k} w_k\right\} \\ &= -m^{3-\alpha}\sum_{n=2}^{\infty}\binom{3-\alpha}{2n}\left(2^{2n+1}-8\right)\frac{1}{m^{2n}} < 0, \quad \alpha \in (1,2). \end{aligned}$$

It should be noted that for odd k, $w_k = 0$, and for even k, $w_k > 0$, and thus $b_{i,j} < 0$.

(II) We next prove $b_{i,i+1} < 0$ or $\widetilde{b}_{i,i+1} < 0$. Since $-7 - 3^{3-\alpha} + 2^{5-\alpha} < 0$ in $b_{i,i+1}$, we just need to check that the first two terms of $\widetilde{b}_{i,i+1}$ are less than zero since the last two terms of $\widetilde{b}_{i,i+1}$ can be discussed in a similar manner by exploiting a symmetry structure. Let $x = \frac{1}{i} \in (0,1]$ with $i = 1, 2, \ldots, N-1$. Setting

$$d(x) = \frac{2}{x}\left((1+x)^{3-\alpha} - 1\right) - (3-\alpha)\left((1+x)^{2+\alpha} + 1\right),$$

we deduce

$$\begin{aligned} d(x) &= \sum_{n=1}^{\infty}(3-\alpha)(2-\alpha)\ldots(2-\alpha-n)\left(\frac{2}{(n+2)!} - \frac{1}{(n+1)!}\right)x^{n+1} \\ &= \sum_{n=1}^{\infty} g_n(x)\frac{x^{2n}}{(n+1)!} > 0, \ x \in (0,1], \end{aligned}$$

where the last equality follows of the equation above when combining the coefficients of x^{2n} and x^{2n+1} and where it is easy to check that $g_n(x)$ is strictly positive, with

$$g_n(x) = (3-\alpha)(2-\alpha)\dots(3-\alpha-2n)\left(2-(2n+1)+(2-\alpha-2n)\left(\frac{2}{2n+2}-1\right)x\right).$$

Hence, using (2.9) and the positivity of $d(x)$, we obtain

$$\begin{aligned}
&-2\left((i+1)^{3-\alpha}-i^{3-\alpha}\right)+(3-\alpha)\left((i+1)^{2-\alpha}+i^{2-\alpha}\right)\\
&\quad= -i^{2-\alpha}\left[2i\left(\left(1+\frac{1}{i}\right)^{3-\alpha}-1\right)-(3-\alpha)\left(\left(1+\frac{1}{i}\right)^{2+\alpha}+1\right)\right]\\
&\quad= -i^{2-\alpha}d(x)<0.
\end{aligned}$$

To conclude, we have $\widetilde{d}(N-i)<0$, which implies $\widetilde{b}_{i,i+1}<0$. Using $-7-3^{3-\alpha}+2^{5-\alpha}<0$ and (2.9), we deduce $b_{i,i+1}<0$.

(III) To prove $\widetilde{b}_{i,i}<0$, we need to verify

$$2\left[(i+1)^{3-\alpha}-2(3-\alpha)i^{2-\alpha}-(i-1)^{3-\alpha}\right]=2i^{2-\alpha}\widetilde{p}(i)<0,$$

with $\widetilde{p}(i)=i\left(1+\frac{1}{i}\right)^{3-\alpha}-i\left(1-\frac{1}{i}\right)^{3-\alpha}-2(3-\alpha)$. Let $x=\frac{1}{i}\in(0,1]$; we can also prove $p(x)=\frac{1}{x}\left((1+x)^{3-\alpha}-(1-x)^{3-\alpha}\right)-2(3-\alpha)<0$ since

$$p(x)=\sum_{n=1}^{\infty}q_n x^{2n},\ x\in(0,1],$$

with $q_n=\frac{2(3-\alpha)(2-\alpha)\dots(3-\alpha-2n)}{(2n+1)!}<0$.

(IV) Finally, we prove $\sum\limits_{j=1}^{N-1} b_{i,j}>0$. According to $\sum_{j=1}^{N-1}\phi_j(x)=1-\phi_0(x)-\phi_N(x)$ and (2.6)–(2.9), there exists

$$\begin{aligned}
\sum_{j=1}^{N-1}a_{i,j} &= \frac{\kappa_\alpha}{h^{\alpha-1}\Gamma(4-\alpha)}\sum_{j=1}^{N-1}b_{i,j}\\
&= C_\alpha\int_{x_{i-1}}^{x_{i+1}}\phi_i(x)\int_\Omega\frac{\sum_{j=1}^{N-1}\phi_j(x)-\sum_{j=1}^{N-1}\phi_j(y)}{|y-x|^\alpha}dydx\\
&= C_\alpha\int_{x_{i-1}}^{x_{i+1}}\phi_i(x)\int_\Omega\frac{-\phi_0(x)-\phi_N(x)+\phi_0(y)+\phi_N(y)}{|y-x|^\alpha}dydx.
\end{aligned}$$

Thus, we deduce that

$$\sum_{j=1}^{N-1}a_{i,j}=C_\alpha\int_{x_{i-1}}^{x_{i+1}}\phi_i(x)\int_\Omega\frac{\phi_0(y)+\phi_N(y)}{|y-x|^\alpha}dydx>0,\ i=2,3,\dots,N-2,$$

and

$$\sum_{j=1}^{N-1} a_{i,j} = C_\alpha \int_{x_{i-1}}^{x_{i+1}} \phi_i(x) \int_\Omega \frac{-\phi_0(x) + \phi_0(y) + \phi_N(y)}{|y-x|^\alpha} dydx$$
$$> C_\alpha \left[\int_{x_{i-1}}^{x_i} \phi_i(x) \int_\Omega \frac{-\phi_0(x) + \phi_0(y)}{|y-x|^\alpha} dydx + \int_{x_i}^{x_{i+1}} \phi_i(x) \int_\Omega \frac{\phi_0(y)}{|y-x|^\alpha} dydx \right]$$
$$> C_\alpha \int_{x_{i-1}}^{x_i} \phi_i(x) \int_\Omega \frac{-\phi_0(x) + \phi_0(y)}{|y-x|^\alpha} dydx = \frac{C_\alpha h^{1-\alpha}}{(2-\alpha)(3-\alpha)} > 0, \quad i = 1.$$

Similarly, we have $\sum_{j=1}^{N-1} a_{i,j} > 0,\ \ i = N-1$. Since $\sum_{j=1}^{N-1} b_{i,j} > 0$ and $b_{i,j} < 0,\ j \neq i$, it yields $b_{i,i} > \sum\limits_{j \neq i} |b_{i,j}| > 0$. The proof is completed. ◻

LEMMA 4.4. *Let A_h be defined by (2.8) with $1 < \alpha < 2$ and D_h be the diagonal of A_h. Then*

$$1 \leq \lambda_{\max}(D_h^{-1} A_h) < 2.$$

Proof. Since $D_h^{-1/2} \left(D_h^{-1/2} A_h D_h^{-1/2} \right) D_h^{1/2} = D_h^{-1} A_h$, we have that $D_h^{-1/2} A_h D_h^{-1/2}$ and $D_h^{-1} A_h$ are similar, i.e., $\lambda_{\max}(D_h^{-1} A_h) = \lambda_{\max}(D_h^{-1/2} A_h D_h^{-1/2})$. Denote $C_h = D_h^{-1/2} A_h D_h^{-1/2}$ with $(C_h)_{i,j} = c_{i,j}$ and $M_h = D_h^{-1} A_h$ with $(M_h)_{i,j} = m_{i,j}$. Using Lemma 4.3 and (2.8), we obtain

$$r_i := \sum_{j \neq i} |m_{i,j}| < m_{i,i} = 1, \quad i = 1, 2, \ldots, N-1.$$

From the Gerschgorin circle theorem [26, p. 388], the eigenvalues of M_h are in the disks centered at $m_{i,i}$ with radius r_i; i.e., the eigenvalues λ of the matrix M_h satisfy

$$|\lambda - m_{i,i}| \leq r_i,$$

which yields $\lambda_{\max}(M_h) = \lambda_{\max}(D_h^{-1} A_h) \leq m_{i,i} + r_i < 2 m_{i,i} = 2$.

On the other hand, using the Rayleigh theorem [26, p. 235], i.e.,

$$\lambda_{\max}(C_h) = \max_{x \neq 0} \frac{x^{\mathrm{T}} C_h x}{x^{\mathrm{T}} x} \quad \forall x \in \mathcal{R}^{n_q},$$

if we take $x = [0, \ldots, 0, 1, 0, \ldots, 0]^{\mathrm{T}}$, it means that

$$\lambda_{\max}(C_h) \geq \frac{x^{\mathrm{T}} C_h x}{x^{\mathrm{T}} x} = c_{i,i} = 1.$$

It yields

$$1 \leq \lambda_{\max}(D_h^{-1} A_h) < 2.$$

The proof is completed. ◻

THEOREM 4.5. *Let $A_q := A_h$ be defined by (2.8) with $1 < \alpha < 2$. Then K_q satisfies (4.1), and the convergence factor of the TGM satisfies*

$$\|K_q T^q\|_{A_q} \leq \sqrt{1 - 2\eta/5} < 1,$$

where $\eta \leq \omega(2 - \omega\eta_0)$ with $0 < \omega < 2/\eta_0$, $\eta_0 < 2$.

Proof. From Lemma 4.4, we obtain $\lambda_{\max}(D_q^{-1}A_q) \le \eta_0 < 2$. Taking $0 < \omega < 2/\eta_0$, $\eta \le \omega(2-\omega\eta_0)$ and using Lemma 4.1, we conclude that K_J satisfies (4.1). Next we prove that (4.2) holds; i.e., we need to find a closed form of the constant κ for A_q applied to Lemma 4.2. Let $\nu_0 = \nu_{n_q+1} = 0$ and

$$\nu^q = [\nu_1, \nu_2, \dots, \nu_{n_q}]^{\mathrm{T}} \in \mathcal{R}^{n_q}, \quad \nu^{q-1} = [\nu_2, \nu_4, \dots, \nu_{n_q-1}]^{\mathrm{T}} \in \mathcal{R}^{n_{q-1}},$$

with $n_q = 2^q - 1$. From [20, 33], we have

$$\|\nu^q - I_{q-1}^q \nu^{q-1}\|^2 = \sum_{i=0}^{n_{q-1}} \left(\nu_{2i+1} - \frac{\nu_{2i}+\nu_{2i+2}}{2}\right)^2.$$

Using (2.8), (2.9), and Lemma 4.3, there exists

$$\begin{aligned}\|\nu^q - I_{q-1}^q \nu^{q-1}\|_{D_q}^2 &= \sum_{i=0}^{n_{q-1}} a_{2i+1,2i+1}\left(\nu_{2i+1} - \frac{\nu_{2i}+\nu_{2i+2}}{2}\right)^2 \\ &\le a_0 \sum_{i=0}^{n_{q-1}} \left(\nu_{2i+1} - \frac{\nu_{2i}+\nu_{2i+2}}{2}\right)^2 \le a_0 \sum_{i=1}^{n_q} \left(\nu_i^2 - \nu_i\nu_{i+1}\right)^2,\end{aligned}$$

with $a_0 = \frac{\kappa_\alpha}{h^{\alpha-1}\Gamma(4-\alpha)}\left(8 - 2^{4-\alpha}\right)$.

Let the symmetric positive definite matrix $L_{n_q} = \mathrm{tridiag}(-1, 2, -1)$ be the $n_q \times n_q$ one-dimensional discrete Laplacian. From (2.8) and (2.9), we have $A_q = -a_1 L_{n_q} + A_{\mathrm{rest}}$ with $a_1 = \frac{\kappa_\alpha}{h^{\alpha-1}\Gamma(4-\alpha)}\left(-7 - 3^{3-\alpha} + 2^{5-\alpha}\right) < 0$. Using Lemma 4.3, it yields that A_{rest} is also the symmetric positive definite matrix that is diagonally dominant. Hence,

$$\|\nu^q\|_{A_q}^2 = (A_q\nu^q, \nu^q) \ge (-a_1 L_{n_q}\nu^q, \nu^q) = -2a_1 \sum_{i=1}^{n_q} \left(\nu_i^2 - \nu_i\nu_{i+1}\right)^2 \quad \forall \nu^q \in \mathcal{R}^{n_q}.$$

According to the above equations, we find

$$\|\nu^q - I_{q-1}^q \nu^{q-1}\|_{D_q}^2 \le a_0 \sum_{i=1}^{n_q} \left(\nu_i^2 - \nu_i\nu_{i+1}\right)^2 \le \kappa \|\nu^q\|_{A_q}^2,$$

with $\kappa = -\frac{a_0}{2a_1} \in (1, \frac{5}{2})$. The proof is completed. ∎

5. Convergence of TGM and full MGM for (2.10): A non–diagonally dominant system. Although there are still questions regarding the best ways to define the coarsening and interpolation operators when the stiffness matrix is far from being weakly diagonally dominant [42], here the simple (traditional) restriction operator and prolongation operator are employed for such algebraic systems. A reason for the latter choice can be found in subsection 2.3, where it is shown that the ill-conditioning of the involved coefficient matrix is due to a vector in low frequency (the vector of all ones). We recall that the standard discrete Laplacian is ill-conditioned only in low frequencies since its spectral symbol $f(\theta) = 2 - 2\cos(\theta)$ has a unique zero at $\theta = 0$ (see [5]), and therefore it is no surprise that the same multigrid ingredients are effective also in our context.

5.1. Convergence of TGM for (2.10): A non–diagonally dominant system. We now start to prove the convergence of TGM for (2.10).

LEMMA 5.1. *Let A_h be defined by* (2.11) *and D_h be the diagonal of A_h. Then*

$$1 \leq \lambda_{\max}(D_h^{-1}A_h) < 3.$$

Proof. From (2.11), we obtain

$$r_i := \sum_{j\neq i} |a_{i,j}| = h^2 \sum_{j \neq i} |b_{i,j}| < 2h^2 b_{i,i} = 2a_{i,i}.$$

From the Gerschgorin circle theorem [26, p. 388], the eigenvalues of A_h are in the disks centered at $a_{i,i}$ with radius r_i; i.e., the eigenvalues λ of the matrix A_h satisfy

$$|\lambda - a_{i,i}| \leq r_i,$$

which yields $\lambda_{\max}(A_h) \leq a_{i,i} + r_i < 3a_{i,i}$. By the same way as Lemma 4.4, we have $\lambda_{\max}(A_h) \geq a_{i,i}$. The proof is completed. ∎

LEMMA 5.2. *Let $B = B_h$ be defined by* (2.11) *and $L_{N-1} = \mathrm{tridiag}(-1, 2, -1)$ be the $(N-1) \times (N-1)$ one-dimensional discrete Laplacian. Then*

$$H := B - \frac{N}{12} L_{N-1}$$

is a positive semidefinite matrix. In particular, H is singular for N even, and it is positive definite for N odd.

Proof. The matrix H can be written as

$$H = \frac{N}{4} \begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{bmatrix}_{(N-1)\times(N-1)} - \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 \end{bmatrix}_{(N-1)\times(N-1)}$$

or more compactly

$$H = \frac{N}{4} T_{N-1}(g(\theta)) - ee^{\mathrm{T}}, \qquad g(\theta) = 2 + 2\cos(\theta), \tag{5.1}$$

with $e = [1, \ldots, 1]^{\mathrm{T}}$ being the vector of all ones of size $N - 1$.

We next prove $(Hx, x) \geq 0 \; \forall x = [x_1, x_2, \ldots, x_{N-1}]^{\mathrm{T}}$. Using the Cauchy–Schwarz inequality with $x_0 = x_N = 0$, we have

$$\begin{aligned} (Hx, x) &= \frac{N}{4} \left(2 \sum_{i=1}^{N-1} x_i^2 + 2 \sum_{i=1}^{N-2} x_i x_{i+1} \right) - \left(\sum_{i=1}^{N-1} x_i \right)^2 \\ &= \frac{1}{4} \sum_{i=0}^{N-1} 1^2 \cdot \sum_{i=0}^{N-1} (x_i + x_{i+1})^2 - \left(\sum_{i=1}^{N-1} x_i \right)^2 \\ &\geq \frac{1}{4} \left(\sum_{i=0}^{N-1} (x_i + x_{i+1}) \right)^2 - \left(\sum_{i=1}^{N-1} x_i \right)^2 = 0. \end{aligned}$$

For proving the singularity of H for N even and its positive definiteness for N odd, we proceed as in subsection 2.3. First, we observe that the eigenvalues of $T_{N-1}(g(\theta))$ belong to the open set $(0,4)$ since $0=\min g(\theta)$, $4=\max g(\theta)$ and $g(\theta)=2+2\cos(\theta)$. Thus, $T_{N-1}(g(\theta))$ is invertible and

$$H=\frac{N}{4}T_{N-1}(g(\theta))\left[I_{N-1}-\frac{4}{N}T_{N-1}^{-1}(g(\theta))ee^{\mathrm{T}}\right]. \tag{5.2}$$

Consequently, by the Binet theorem, we have

$$\det(H)=\det(Y)\det\left(I_{N-1}-\frac{4}{N}T_{N-1}^{-1}(g(\theta))ee^{\mathrm{T}}\right)=\det(Y)\left(1-\frac{4}{N}e^{\mathrm{T}}T_{N-1}^{-1}(g(\theta))e\right), \tag{5.3}$$

with $Y=\frac{N}{4}T_{N-1}(g(\theta))$.

Now there is a basic similarity relation between $T_{N-1}(g(\theta))$ and the discrete Laplacian $L_{N-1}=\mathrm{tridiag}(-1,2,-1)=T_{N-1}(2-2\cos(\theta))$ since

$$T_{N-1}(g(\theta))=DL_{N-1}D, \qquad D=D^{-1}=\mathrm{diag}((-1)^j : 1\le j\le N-1).$$

Consequently, by (5.3) and by the positivity of $\det(Y)$, the sign of $\det(H)$ depends on the quantity

$$\phi(N)=1-\frac{4x}{N}, \qquad x=e^{\mathrm{T}}T_{N-1}^{-1}(g(\theta))e=e^{\mathrm{T}}DL_{N-1}^{-1}De, \tag{5.4}$$

that is,

$$\det(H)=\det(Y)\phi(N).$$

The quantities in (5.4) have an explicit expression since the inverse of the discrete Laplacian is known, and in particular, we have $\left(L_{N-1}^{-1}\right)_{r,c}=t_r^{(c)}$, $\left(T_{N-1}^{-1}(g(\theta))\right)_{r,c}=t_r^{(c)}(-1)^{r+c}$ with

$$t_r^{(c)}=\frac{(N-c)r}{N}, \quad r=1,\ldots,c, \tag{5.5}$$

$$t_r^{(c)}=\frac{(N-r)c}{N}, \quad r=c+1,\ldots,N-1. \tag{5.6}$$

Finally, using the latter explicit values in (5.4)–(5.6), we obtain

$$x=\begin{cases}\frac{N^2}{4N}=\frac{N}{4}, & N \text{ even},\\ \frac{N^2-1}{4N}, & N \text{ odd},\end{cases} \tag{5.7}$$

that is,

$$\phi(N)=1-\frac{4x}{N}=\begin{cases}0, & N \text{ even},\\ \frac{1}{N^2}, & N \text{ odd},\end{cases} \tag{5.8}$$

and the proof is concluded. ∎

THEOREM 5.3. *Let $A_q := A_h$ be defined by (2.11). Then K_q satisfies (4.1), and the convergence factor of the TGM satisfies*

$$\|K_qT^q\|_{A_q}\le\sqrt{1-\eta/4}<1,$$

where $\eta\le\omega(2-\omega\eta_0)$, $0<\omega<2/\eta_0$, and $\eta_0<3$.

Proof. From Lemma 5.1, we have $\lambda_{\max}(D_q^{-1}A_q) \leq \eta_0 < 3$. Taking $0 < \omega < 2/\eta_0$, $\eta \leq \omega(2-\omega\eta_0)$ and using Lemma 4.1, we conclude that K_J satisfies (4.1). Next we prove that (4.2) holds; i.e., we need to find a closed form of the constant κ for A_q applied to Lemma 4.2. Let

$$\nu^q = [\nu_1, \nu_2, \dots, \nu_{n_q}]^{\mathrm{T}} \in \mathcal{R}^{n_q}, \quad \nu^{q-1} = [\nu_2, \nu_4, \dots, \nu_{n_q-1}]^{\mathrm{T}} \in \mathcal{R}^{n_{q-1}},$$

and $\nu_0 = \nu_{n_q+1} = 0$ with $n_q = 2^q - 1$. From [20, 33] and (2.11), we have

$$\|\nu^q - I_{q-1}^q \nu^{q-1}\|_{D_q}^2 \leq a_0 \sum_{i=1}^{n_q} \left(\nu_i^2 - \nu_i\nu_{i+1}\right)^2 = \frac{a_0}{2}(L_{n_q}\nu^q, \nu^q),$$

with $a_0 = h^2\left(\frac{2N}{3} - 1\right)$, $N = n_q + 1$, and $L_{n_q} = \mathrm{tridiag}(-1, 2, -1)$.

Using Lemma 5.2 and (2.11), we infer

$$\|\nu^q\|_{A_q}^2 = (A_q\nu^q, \nu^q) = h^2(B_h\nu^q, \nu^q) \geq h^2\frac{N}{12}(L_{n_q}\nu^q, \nu^q).$$

According to the above equations, we conclude

$$\|\nu^q - I_{q-1}^q \nu^{q-1}\|_{D_q}^2 \leq \frac{a_0}{2}(L_{n_q}\nu^q, \nu^q) \leq \frac{a_0}{2}\frac{12}{h^2N}\|\nu^q\|_{A_q}^2 < 4\|\nu^q\|_{A_q}^2,$$

and the proof is completed. ∎

5.2. Convergence of the full MGM for (2.10): A non–diagonally dominant system. We extend the convergence results of TGM given in the above subsection to the full MGM. To the best of our knowledge, it is the first time that a convergence analysis of the full MGM for the dense matrix case has been studied.

LEMMA 5.4. *Let $B^{(1)} = B_h = \{b_{i,j}^{(1)}\}_{i,j=1}^{N-1}$ with $b_{i,j}^{(1)} = b_{|i-j|}^{(1)}$ be given in (2.11) and $D_{(k)}$ be the diagonal of the matrix $B^{(k)}$, where $B^{(k)} = L_h^H B^{(k-1)} L_H^h$ with $L_h^H = 4I_k^{k-1}$ and $L_H^h = (L_h^H)^{\mathrm{T}}$. Then*

$$1 \leq \lambda_{\max}\left(D_{(k)}^{-1}B^{(k)}\right) < 3, \quad 1 \leq k \leq q,$$

with q being the total number of levels.

Proof. From (2.11), we have

$$B_h = NI - \frac{N}{6}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(N-1)\times(N-1)} - \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 \end{bmatrix}_{(N-1)\times(N-1)},$$

where I is an identity matrix and $N = 2^q$ (i.e., $n_q = N - 1$). Using Lemma 3.1, it yields

(5.9)

$$B^{(k)} = N\begin{bmatrix} 4C_k + 2^{k-1} & C_k & & & \\ C_k & 4C_k + 2^{k-1} & C_k & & \\ & \ddots & \ddots & \ddots & \\ & & C_k & 4C_k + 2^{k-1} & C_k \\ & & & C_k & 4C_k + 2^{k-1} \end{bmatrix}_{(\frac{N}{2^{k-1}}-1)\times(\frac{N}{2^{k-1}}-1)}$$

$$- \frac{N}{6}\begin{bmatrix} 2^k & -2^{k-1} & & & \\ -2^{k-1} & 2^k & -2^{k-1} & & \\ & \ddots & \ddots & \ddots & \\ & & -2^{k-1} & 2^k & -2^{k-1} \\ & & & -2^{k-1} & 2^k \end{bmatrix}_{(\frac{N}{2^{k-1}}-1)\times(\frac{N}{2^{k-1}}-1)}$$

$$- 16^{k-1}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{(\frac{N}{2^{k-1}}-1)\times(\frac{N}{2^{k-1}}-1)} .$$

Denote $B^{(k)} = \{b_{i,j}^{(k)}\}_{i,j=1}^{\frac{N}{2^{k-1}}-1}$ with $b_{i,j}^{(k)} = b_{|i-j|}^{(k)}$. Thus, we obtain

$$\begin{aligned} b_0^{(k)} &= \frac{2^{3k-2}}{3}N - 2^{4k-4}, \quad b_1^{(k)} = \frac{2^{3k-4}}{3}N - 2^{4k-4}, \\ b_l^{(k)} &= -2^{4k-4}, \quad 2 \le l \le \frac{N}{2^{k-1}} - 2, \end{aligned} \tag{5.10}$$

so that

$$\begin{aligned} &2b_0^{(k)} - 2b_1^{(k)} - \frac{N}{2^{k-1}}2^{4k-4} = 0 \ \text{ if } \ b_1^{(k)} \ge 0, \\ &b_0^{(k)} - (-2b_1^{(k)}) - \left(\frac{N}{2^{k-1}} - 3\right) 2^{4k-4} = 0 \ \text{ if } \ b_1^{(k)} \le 0. \end{aligned}$$

In a word, there exists

$$r_i^{(k)} \coloneqq \sum_{j\ne i} |b_{i,j}^{(k)}| < 2b_{i,i}^{(k)}.$$

By following the same steps as in Lemma 5.1, we have $b_{i,i}^{(k)} \le \lambda_{\max}(B^{(k)}) < 3b_{i,i}^{(k)}$. The proof is completed. ∎

LEMMA 5.5. *Let $B^{(k)}$ be defined by (5.9) and $L_{\frac{N}{2^{k-1}}-1} = \mathrm{tridiag}(-1, 2, -1)$ be the $(\frac{N}{2^{k-1}} - 1) \times (\frac{N}{2^{k-1}} - 1)$ one-dimensional discrete Laplacian. Then*

$$H^{(k)} \coloneqq B^{(k)} - \frac{2^{3k-5}}{3}NL_{\frac{N}{2^{k-1}}-1}, \quad 1 \le k \le q, \ N = 2^q,$$

is a positive semidefinite matrix. Here q is a total number of levels.

Proof. Using $C_k + 2^{k-1}/6 = 2^{3k-4}/3$, we can rewrite (5.9) as

$$B^{(k)} = 2^{3k-3}NI - \frac{2^{3k-4}}{3}N \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{\left(\frac{N}{2^{k-1}}-1\right)\times\left(\frac{N}{2^{k-1}}-1\right)}$$

$$- 2^{4k-4}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{\left(\frac{N}{2^{k-1}}-1\right)\times\left(\frac{N}{2^{k-1}}-1\right)},$$

where I is an identity matrix. Thus,

$$2^{4-4k}H^{(k)} = \frac{\widetilde{N}}{4}\begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{bmatrix}_{(\widetilde{N}-1)\times(\widetilde{N}-1)} - \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 \end{bmatrix}_{(\widetilde{N}-1)\times(\widetilde{N}-1)},$$

with $\widetilde{N} = \frac{N}{2^{k-1}}$. By following the same steps as in Lemma 5.2, the desired result is obtained. ∎

THEOREM 5.6. *Let $A_q := A_h$ be defined by (2.11). Then K_k satisfies (4.1), and the convergence factor of the full MGM satisfies*

$$\|K_k T^k\|_{A_k} \le \sqrt{1-\eta/4} < 1, \quad 1 \le k \le q,$$

where $\eta \le \omega(2-\omega\eta_0)$ with $0<\omega<2/\eta_0$ and $\eta_0 < 3$.

Proof. From Lemma 5.4 and (2.11), we have $\lambda_{\max}(D_k^{-1}A_k) \le \eta_0 < 3$. Taking $0<\omega<2/\eta_0$, $\eta \le \omega(2-\omega\eta_0)$ and using Lemma 4.1, we conclude that K_k satisfies (4.1). Next we prove that (4.2) holds; i.e., we need to find a closed form of the constant κ for A_k applied to Lemma 4.2. Let $A_k = A^{(q-k+1)}$, $A^{(k)} = L_h^H A^{(k-1)} L_H^h$ with $L_h^H = 4I_k^{k-1}$ and $L_H^h = (L_h^H)^{\mathrm{T}}$, and $D^{(k)}$ be the diagonal of $A^{(k)}$. Let n_k be the size of the matrix $A^{(k)}$ and

$$\nu^k = [\nu_1, \nu_2, \dots, \nu_{n_k}]^{\mathrm{T}} \in \mathcal{R}^{n_k}, \quad \nu^{k-1} = [\nu_2, \nu_4, \dots, \nu_{n_k-1}]^{\mathrm{T}} \in \mathcal{R}^{n_{k-1}},$$

with $\nu_0 = \nu_{n_k+1} = 0$. From [20, 33] and (2.11), (5.10), we have

$$\|\nu^k - I_{k-1}^k \nu^{k-1}\|_{D^{(k)}}^2 \le a_0^{(k)} \sum_{i=1}^{n_k} \left(\nu_i^2 - \nu_i\nu_{i+1}\right)^2 = \frac{a_0^{(k)}}{2}(L_{n_k}\nu^k, \nu^k),$$

with $a_0^{(k)} = h^2 b_0^{(k)} = h^2\left(\frac{2^{3k-2}}{3}N - 2^{4k-4}\right)$ and $L_{n_k} = \mathrm{tridiag}(-1,2,-1)$.

Using (2.11), (5.9), (5.10), and Lemma 5.5, we find

$$\|\nu^k\|_{A^{(k)}}^2 = (A^{(k)}\nu^k, \nu^k) = h^2(B^{(k)}\nu^k, \nu^k) \ge h^2\frac{2^{3k-5}}{3}N(L_{n_k}\nu^k, \nu^k).$$

According to the above equations, we deduce

$$\|\nu^k - I_{k-1}^k \nu^{k-1}\|_{D^{(k)}}^2 \leq \frac{a_0^{(k)}}{2}(L_{n_k}\nu^k, \nu^k) \leq \frac{a_0^{(k)}}{2}\frac{3}{h^2 2^{3k-5}N}\|\nu^k\|_{A^{(k)}}^2 < 4\|\nu^k\|_{A^{(k)}}^2,$$

and the proof is completed. ∎

6. Numerical results. We employ the V-cycle MGM described in Algorithm 1 to solve the steady-state nonlocal problems (2.1). The stopping criterion is taken as

$$\frac{\|r^{(i)}\|}{\|r^{(0)}\|} < 10^{-10} \quad \text{for} \quad (2.7), \quad \frac{\|r^{(i)}\|}{\|r^{(0)}\|} < 10^{-13} \quad \text{for} \quad (2.10),$$

where $r^{(i)}$ is the residual vector after i iterations and the number of iterations $(m_1, m_2) = (1, 2)$ and $(\omega_{pre}, \omega_{post}) = (1, 1)$ for (2.7) and $(\omega_{pre}, \omega_{post}) = (1/2, 1)$ for (2.10). In all tables, N denotes the number of spatial grid points, the numerical errors are measured by the l_∞ (maximum) norm, "Rate" denotes the convergence orders, "CPU" denotes the total CPU time in seconds (s) for solving the resulting discretized systems, and "Iter" denotes the average number of iterations required to solve a general linear system $A_h u_h = f_h$.

All numerical experiments are programmed in MATLAB, and the computations are carried out on a PC with the following configuration: Intel Core i5-3470 3.20 GHz and 8 GB RAM and a 64-bit Windows 7 operating system.

Example 6.1 (a Toeplitz-plus-tridiagonal system). Consider the steady-state nonlocal problem (2.7) on a finite domain $0 < x < b$, $b = 2$. The exact solution of the equation is $u(x) = x^2(b-x)^2$ and the source function

$$\begin{aligned} f(x) = &- \frac{\kappa_\alpha \alpha(\alpha-5)(-\alpha^2-5\alpha-10)}{\Gamma(5-\alpha)}\left(x^{4-\alpha} + (b-x)^{4-\alpha}\right) \\ &+ \frac{2b\kappa_\alpha\alpha(\alpha^2-6\alpha+11)}{\Gamma(4-\alpha)}\left(x^{3-\alpha} + (b-x)^{3-\alpha}\right) \\ &- \frac{b^2\kappa_\alpha\alpha(3-\alpha)}{\Gamma(3-\alpha)}\left(x^{2-\alpha} + (b-x)^{2-\alpha}\right). \end{aligned}$$

Table 6.1 shows that the numerical scheme has second-order accuracy and the computation cost is almost $\mathcal{O}(N\log N)$ operations.

Example 6.2 (a non–diagonally dominant system). Consider the steady-state nonlocal problem (2.10) on a finite domain $0 < x < b$, $b = 2$. The exact solution of the equation is $u(x) = x^2(b-x)^2$ and the source function

$$f(x) = bx^2(b-x)^2 - \frac{1}{30}b^5.$$

Table 6.1

Using Galerkin approach $A_{k-1} = I_k^{k-1} A_k I_{k-1}^k$ computed by Lemmas 3.1 *and* 3.2 *to solve the resulting systems of (*2.7*).*

N	$\alpha = 1.3$	Rate	Iter	CPU	$\alpha = 1.7$	Rate	Iter	CPU
2^9	1.6294e-05		30	0.17 s	1.3629e-05		79	0.22 s
2^{10}	4.1063e-06	1.9884	31	0.31 s	3.5307e-06	1.9487	79	0.39 s
2^{11}	1.0284e-06	1.9974	33	0.60 s	9.0793e-07	1.9593	78	0.71 s
2^{12}	2.5718e-07	1.9996	35	1.17 s	2.3572e-07	1.9455	78	1.34 s

Table 6.2
Using Galerkin approach $A_{k-1} = I_k^{k-1} A_k I_{k-1}^k$ *computed by Lemma* 3.1 *or* (3.2) *to solve the resulting systems of* (2.10)*.*

N		Rate	Iter	CPU	N		Rate	Iter	CPU
2^{11}	9.5325e-07		83	0.38 s	2^{14}	1.4910e-08	1.9991	86	2.61 s
2^{12}	2.3837e-07	1.9997	84	0.70 s	2^{15}	3.7396e-09	1.9953	86	4.90 s
2^{13}	5.9603e-08	1.9997	85	1.33 s	2^{16}	9.6707e-10	1.9512	87	9.76 s

Table 6.2 shows that the numerical scheme has second-order accuracy and the computation cost is almost $\mathcal{O}(N\log N)$ operations.

Remark 6.1. The examples chosen for the numerical experiments showed that the CPU time ratio is less than $\mathcal{O}(N\log N)$: This implies that our algorithm works very effectively for the present nonlocal model (better than methods based on the fast Fourier transform).

7. Conclusions. In this paper, we considered the solutions of Toeplitz-plus-tridiagonal systems, which are far from being weakly diagonally dominant and which arise from nonlocal problems when using linear finite element approximations. We provided the convergence rate of the TGM for nonlocal problems with the fractional Laplace kernel, which is a Toeplitz-plus-tridiagonal system. In this specific context, we answered the question of how to define the coarsening and interpolation operators when the stiffness matrix is non–diagonally dominant [42]. The simple (traditional) restriction operator and prolongation operator are employed for such algebraic systems so that the entries of the sequence of subsystems are explicitly determined on different levels. In the case of the constant (Laplacian-style) kernel, we gave a quite accurate spectral analysis and, based on that, on the structure analysis, and on the computation of the characteristic values at different levels, we extended the TGM convergence results to the full MGM.

For the future, at least two questions arise concerning the analysis of the spectral features and the study of the MGM convergence analysis in the case of a general fractional Laplace kernel.

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