

Eigenvalue Isogeometric Approximations Based on B-Splines: Tools and Results



Sven-Erik Ekström and Stefano Serra-Capizzano

Abstract In this note, we focus on the spectral analysis of large matrices coming from isogeometric approximations based on B-splines of the eigenvalue problem

$$-(a(x)u'(x))' = \lambda b(x)u(x), \quad x \in (0, 1),$$

where $u(0)$ and $u(1)$ are given. When considering the collocation case, global distribution results for the eigenvalues are available in the literature, despite the nonsymmetry of the related matrices. Here we complement such results by providing precise estimates for the extremal eigenvalues and hence for the spectral conditioning of the resulting matrices. In the Galerkin setting, the matrices are symmetric and positive definite and a more complete analysis has been conducted in the literature. In the latter case we furnish a further procedure that gives a highly accurate estimate of all the eigenvalues, starting from the knowledge of the spectral distribution symbol. The techniques involve dyadic decomposition arguments, tools from the theory of generalized locally Toeplitz sequences, and basic extrapolation methods.

1 Introduction

In this note we consider the approximation of one-dimensional elliptic eigenvalue problems by using an isogeometric either Galerkin or collocation technique with B-splines [5]. We are interested in the eigenvalues of the large matrices stemming from the considered approximation processes. In particular, we address the problem of estimating the extremal eigenvalues and of providing efficient numerical procedures for computing a good approximation of all the eigenvalues.

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In this direction, it has recently been proven that the resulting sequence of matrices, indexed with respect to the matrix size, have a canonical distribution (see [15, 17] and references therein), by using the theory of Generalized Locally Toeplitz (GLT) sequences [16].

We recall that every GLT sequence has an associated function, called the symbol, and that the uniform sampling of the symbol provides an asymptotic approximation of all the eigenvalues, if every matrix of the GLT sequence is Hermitian as it happens in our setting. However, in general the approximation is quite poor and recently some extrapolation techniques have been devised (see [9] and references therein). In the constant coefficient setting, when considering the Galerkin approach, the results presented in [10] are impressive in the sense that machine precision is obtained with very low computational cost, while for the variable coefficients further improvements are needed (see also [17]).

In the current note, when considering the Galerkin setting and variable coefficients, we propose a further numerical scheme for the computation of all the spectrum of large matrices by using the numerical computation of the eigenvalues for small matrices, underlying asymptotic expansions, and extrapolation methods as those in [9]. The numerical results are of the same quality as those produced by the best strategy in [17], especially when the problem coefficients are smooth and at least for low frequencies, which are those of highest interest for understanding the nature of the problem.

On the other hand, when dealing with the collocation approximation we obtain large nonsymmetric matrices. However, the GLT machinery can be employed and the symbol is real-valued and nonnegative [7], as in the symmetric positive definite Galerkin setting [15]. Hence we expect that most of the eigenvalues are real or with negligible imaginary part. Here we start the analysis of the collocation case, by describing a technique based on dyadic decompositions, for estimating the extreme eigenvalues and hence the asymptotic (spectral) conditioning of the involved matrix sequences. The analysis contained in Theorem 1 and Corollary 1 confirms that the conditioning grows at most as h^{-2} , h being the fineness parameter, exactly as in the case of the matrices obtained with the Galerkin approximation (see e.g [14]).

Finally, it is not clear if the delicate asymptotic expansions observed in [10, 17] holds also in the collocation setting and indeed this issue will be the subject of future investigations.

2 Preliminaries

In the following we present the notation that we use. In particular we give the definition of eigenvalue distribution, that of rearrangement, and we briefly discuss the informal meaning behind these definitions.

A matrix-sequence is any sequence of the form $\{\mathbf{X}_n\}_n$, where \mathbf{X}_n is a square matrix such that $\text{size}(\mathbf{X}_n) \rightarrow \infty$ as $n \rightarrow \infty$. Let μ_d be the Lebesgue measure in \mathbb{R}^d and let $C_c(\mathbb{C})$ be the space of continuous complex-valued functions with bounded

support defined on \mathbb{C} . If \mathbf{X} is an $m \times m$ matrix, the eigenvalues of \mathbf{X} are denoted by $\lambda_1(\mathbf{X}), \dots, \lambda_m(\mathbf{X})$. If $\mathbf{g} : D \subset \mathbb{R}^d \rightarrow \mathbb{C}^{s \times s}$ is an $s \times s$ matrix-valued function, we say that \mathbf{g} is measurable if its s^2 components $g_{ij} : D \rightarrow \mathbb{C}$, $i, j = 1, \dots, s$, are measurable.

Definition 1 Let $\{\mathbf{X}_n\}_n$ be a matrix-sequence, let $N_n := \text{size}(\mathbf{X}_n)$, and let $\mathbf{g} : D \subset \mathbb{R}^d \rightarrow \mathbb{C}^{s \times s}$ be a measurable $s \times s$ matrix-valued function defined on a set D with $0 < \mu_d(D) < \infty$. We say that $\{\mathbf{X}_n\}_n$ has an (asymptotic) eigenvalue distribution described by \mathbf{g} , and we write $\{\mathbf{X}_n\}_n \sim_\lambda \mathbf{g}$, if

$$\lim_{n \rightarrow \infty} \frac{1}{N_n} \sum_{i=1}^{N_n} F(\lambda_i(\mathbf{X}_n)) = \frac{1}{\mu_d(D)} \int_D \frac{\sum_{i=1}^s F(\lambda_i(\mathbf{g}(y_1, \dots, y_d)))}{s} dy_1 \dots dy_d, \quad \forall F \in C_c(\mathbb{C}),$$

where $\lambda_i(\mathbf{g}(y_1, \dots, y_d))$, $i = 1, \dots, s$, are the eigenvalues of the $s \times s$ matrix $\mathbf{g}(y_1, \dots, y_d)$.

The informal meaning behind the eigenvalue distribution $\{\mathbf{X}_n\}_n \sim_\lambda \mathbf{g}$ is the following: for all sufficiently large n , the eigenvalues of \mathbf{X}_n can be subdivided into s different subsets of approximately the same cardinality; and the eigenvalues belonging to the i th subset (except possibly for $o(N_n)$ outliers) are approximated by the samples of the i th eigenvalue function $\lambda_i(\mathbf{g}(y_1, \dots, y_d))$ over a uniform grid in D (the domain of \mathbf{g}). For example, if $d = 1$, $N_n = ns$ and $D = [a, b]$, then the eigenvalues of \mathbf{X}_n are approximately equal to

$$\lambda_i\left(\mathbf{g}\left(a + j \frac{b-a}{n}\right)\right), \quad j = 1, \dots, n, \quad i = 1, \dots, s,$$

for n large enough. Likewise, if $d = 2$, $N_n = n^2s$ and $D = [a_1, b_1] \times [a_2, b_2]$, then the eigenvalues of \mathbf{X}_n are approximately equal to

$$\lambda_i\left(\mathbf{g}\left(a_1 + j_1 \frac{b_1 - a_1}{n}, a_2 + j_2 \frac{b_2 - a_2}{n}\right)\right), \quad j_1, j_2 = 1, \dots, n, \quad i = 1, \dots, s,$$

for n large enough.

Remark 1 (rearrangement) Let $\mathbf{g} : D \subseteq \mathbb{R}^d \rightarrow \mathbb{C}^{s \times s}$ and suppose that D is a rectangle in \mathbb{R}^d , say $D := [a_1, b_1] \times \dots \times [a_d, b_d]$. We also assume that the eigenvalues $\lambda_1(\mathbf{g}(y_1, \dots, y_d)), \dots, \lambda_s(\mathbf{g}(y_1, \dots, y_d))$ are real for all $(y_1, \dots, y_d) \in D$. For each positive integer r , let \mathcal{G}_r be the uniform grid in D given by

$$\mathcal{G}_r := \left\{ \left(a_1 + \frac{i_1}{r}(b_1 - a_1), \dots, a_d + \frac{i_d}{r}(b_d - a_d) \right) : i_1, \dots, i_d = 1, \dots, r \right\}.$$

Compute the samples of the eigenvalue functions

$$\lambda_1(\mathbf{g}(y_1, \dots, y_d)), \dots, \lambda_s(\mathbf{g}(y_1, \dots, y_d))$$

at the points $(y_1, \dots, y_d) \in \mathcal{G}_r$, sort them in increasing order and put them in a vector $(q_1, q_2, \dots, q_{sr^d})$. Let $\eta_r : [0, 1] \rightarrow \mathbb{R}$ be the piecewise linear non-decreasing function that interpolates the samples $(q_0 := q_1, q_1, q_2, \dots, q_{sr^d})$ over the nodes $(0, \frac{1}{sr^d}, \frac{2}{sr^d}, \dots, 1)$, i.e.,

$$\eta_r\left(\frac{\ell}{sr^d}\right) := q_\ell, \quad \ell = 0, \dots, sr^d.$$

Under certain (normally satisfied) conditions on \mathbf{g} , the function η_r converges (a.e.) as $r \rightarrow \infty$ to a non-decreasing function $\eta : [0, 1] \rightarrow \mathbb{R}$, which is referred to as the rearranged version of \mathbf{g} . What is important about η is that

$$\int_D \frac{\sum_{i=1}^s F(\lambda_i(\mathbf{g}(y_1, \dots, y_d)))}{s} dy_1 \dots dy_d = \int_0^1 F(\eta(t)) dt, \quad \forall F \in C_c(\mathbb{C}).$$

Therefore, if we have $\{\mathbf{X}_n\}_n \sim_\lambda \mathbf{g}$, then we also have $\{\mathbf{X}_n\}_n \sim_\lambda \eta$.

3 Isogeometric Galerkin Discretization of Variable-Coefficient Eigenvalue Problems

Consider the following one-dimensional variable-coefficient eigenvalue problem:

$$\begin{cases} -(a(x)u_j'(x))' = \lambda_j b(x)u_j(x), & x \in \Omega, \\ u_j(x) = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where Ω is an open interval in \mathbb{R} and $a, b \in L^1(\Omega)$ are such that $a, b > 0$ a.e. on Ω . The corresponding weak formulation reads as follows: find eigenvalues $\lambda_j \in \mathbb{R}^+$ and eigenfunctions $u_j \in H_0^1(\Omega)$, for $j = 1, 2, \dots, \infty$, such that, for all $v \in H_0^1(\Omega)$,

$$a(u_j, v) = \lambda_j (b u_j, v),$$

where

$$a(u_j, v) := \int_\Omega a(x)u_j'(x)v'(x)dx, \quad (b u_j, v) := \int_\Omega b(x)u_j(x)v(x)dx.$$

In the isogeometric Galerkin method, we assume that the physical domain Ω is described by a global geometry map $G : [0, 1] \rightarrow \overline{\Omega}$, which is invertible and satisfies $G(\partial([0, 1])) = \partial\overline{\Omega}$. We fix a set of basis functions $\{\varphi_1, \dots, \varphi_{N_n}\}$ defined

on the reference (parametric) domain $[0, 1]$ and vanishing on the boundary $\partial([0, 1])$. We consider the basis functions

$$\psi_i(x) := \varphi_i(G^{-1}(x)) = \varphi_i(t), \quad x = G(t), \quad i = 1, \dots, N_n,$$

which are defined on the physical domain Ω , and we define the approximation space $\mathscr{W}_n := \text{span}(\psi_1, \dots, \psi_{N_n}) \subset H_0^1(\Omega)$. Finally, we find approximations to the exact eigenpairs (λ_j, u_j) , $j = 1, 2, \dots, \infty$, by solving the following Galerkin problem: find $\lambda_{j,n} \in \mathbb{R}^+$ and $u_{j,n} \in \mathscr{W}_n$, for $j = 1, \dots, N_n$, such that, for all $v_n \in \mathscr{W}_n$,

$$\mathbf{a}(u_{j,n}, v_n) = \lambda_{j,n} \mathbf{b}(u_{j,n}, v_n). \quad (2)$$

Assuming that both the exact and numerical eigenvalues are arranged in non-decreasing order, the pair $(\lambda_{j,n}, u_{j,n})$ is taken as an approximation of the pair (λ_j, u_j) for all $j = 1, \dots, N_n$. The numbers $\lambda_{j,n}/\lambda_j - 1$, $j = 1, \dots, N_n$, are referred to as the (relative) eigenvalue errors.

In view of the canonical identification of each function $v_n \in \mathscr{W}_n$ with its coefficient vector with respect to the basis $\{\varphi_1, \dots, \varphi_{N_n}\}$, solving the Galerkin problem (2) is equivalent to solving the generalized eigenvalue problem

$$\mathbf{K}_n(a, G)\mathbf{u}_{j,n} = \lambda_{j,n}\mathbf{M}_n(b, G)\mathbf{u}_{j,n}, \quad (3)$$

where $\mathbf{u}_{j,n}$ is the coefficient vector of $u_{j,n}$ with respect to $\{\varphi_1, \dots, \varphi_{N_n}\}$ and

$$\begin{aligned} \mathbf{K}_n(a, G) &:= [\mathbf{a}(\psi_j, \psi_i)]_{i,j=1}^{N_n} = \left[\int_{\Omega} a(x) \psi_j'(x) \psi_i'(x) dx \right]_{i,j=1}^{N_n} \\ &= \left[\int_0^1 \frac{a(G(t))}{|G'(t)|} \varphi_j'(t) \varphi_i'(t) dt \right]_{i,j=1}^{N_n}, \end{aligned} \quad (4)$$

$$\begin{aligned} \mathbf{M}_n(b, G) &:= [\mathbf{b}(\psi_j, \psi_i)]_{i,j=1}^{N_n} = \left[\int_{\Omega} b(x) \psi_j(x) \psi_i(x) dx \right]_{i,j=1}^{N_n} \\ &= \left[\int_0^1 b(G(t)) |G'(t)| \varphi_j(t) \varphi_i(t) dt \right]_{i,j=1}^{N_n}. \end{aligned} \quad (5)$$

The matrices $\mathbf{K}_n(a, G)$ and $\mathbf{M}_n(b, G)$ are, respectively, the stiffness and mass matrices. Due to our assumption that $a, b > 0$ a.e. on Ω , both $\mathbf{K}_n(a, G)$ and $\mathbf{M}_n(b, G)$ are always symmetric positive definite, regardless of the chosen basis functions $\varphi_1, \dots, \varphi_{N_n}$ and the map G . Moreover, it is clear from (3) that the numerical eigenvalues $\lambda_{j,n}$, $j = 1, \dots, N_n$, are just the eigenvalues of the matrix

$$\mathbf{L}_n(a, b, G) := (\mathbf{M}_n(b, G))^{-1} \mathbf{K}_n(a, G). \quad (6)$$

Therefore, if $a, b \in L^1(\Omega)$ and the basis functions $\varphi_1, \dots, \varphi_{N_n}$ are chosen as the B-splines of degree p and global smoothness C^k defined on the interval $[0, 1]$ partitioned into n equal subintervals (see [17, Sect. 2.1]) and denoted by

$$B_{2,[p,k]}, \dots, B_{n(p-k)+k,[p,k]}, \quad (7)$$

then

$$\mathbf{K}_n(a, G) = \left[\int_0^1 \frac{a(G(t))}{|G'(t)|} B'_{j+1,[p,k]}(t) B'_{i+1,[p,k]}(t) dt \right]_{i,j=1}^{n(p-k)+k-1}, \quad (8)$$

$$\mathbf{M}_n(b, G) = \left[\int_0^1 b(G(t)) |G'(t)| B_{j+1,[p,k]}(t) B_{i+1,[p,k]}(t) dt \right]_{i,j=1}^{n(p-k)+k-1}. \quad (9)$$

From the *-algebra structure of GLT sequences, as proven in [16, 17], we know that

$$\left\{ \frac{1}{n} \mathbf{K}_n(a, G) \right\}_n \sim_\lambda \kappa_{[p,k]}, \quad (10)$$

$$\left\{ n \mathbf{M}_n(b, G) \right\}_n \sim_\lambda \xi_{[p,k]}, \quad (11)$$

$$\left\{ \frac{1}{n^2} \mathbf{L}_n(a, b, G) \right\}_n \sim_\lambda \zeta_{[p,k]}, \quad (12)$$

where, according to Remark 1, $\kappa_{[p,k]}$, $\xi_{[p,k]}$, $\zeta_{[p,k]}$ are the rearranged versions of

$$\frac{a(G(t))}{|G'(t)|} \mathbf{f}_{[p,k]}(\theta), \quad b(G(t)) |G'(t)| \mathbf{h}_{[p,k]}(\theta), \quad \frac{a(G(t))}{b(G(t)) |G'(t)|^2} \mathbf{e}_{[p,k]}(\theta),$$

respectively, with

$$\mathbf{f}_{[p,k]}(\theta) := \mathbf{K}_{[p,k]}^{[0]} + \sum_{\ell=1}^{\eta-1} \left(\mathbf{K}_{[p,k]}^{[\ell]} e^{i\ell\theta} + (\mathbf{K}_{[p,k]}^{[\ell]})^T e^{-i\ell\theta} \right), \quad (13)$$

$$\mathbf{h}_{[p,k]}(\theta) := \mathbf{M}_{[p,k]}^{[0]} + \sum_{\ell=1}^{\eta-1} \left(\mathbf{M}_{[p,k]}^{[\ell]} e^{i\ell\theta} + (\mathbf{M}_{[p,k]}^{[\ell]})^T e^{-i\ell\theta} \right), \quad (14)$$

$$\mathbf{e}_{[p,k]}(\theta) := (\mathbf{h}_{[p,k]}(\theta))^{-1} \mathbf{f}_{[p,k]}(\theta), \quad (15)$$

where the blocks $\mathbf{K}_{[p,k]}^{[\ell]}$ and $\mathbf{M}_{[p,k]}^{[\ell]}$, of size $p-k$, are defined in [17]. The analytical predictions of the eigenvalue errors are obtained through the following sampling procedure:

$$\frac{\lambda_{j,n}}{\lambda_j} - 1 \approx \frac{n^2 \zeta_{[p,k]} \left(\frac{j}{n(p-k)} \right)}{\lambda_j} - 1, \quad j = 1, \dots, m, \quad (16)$$

where $m := \min(n(p - k), n(p - k) + k - 1)$.

However, such a procedure is not completely satisfactory especially in low frequencies (see [17]), when variable coefficients are considered. More precisely, we recall that the smallest eigenvalues are those related to slowly oscillating modes, that is the related eigenvectors belong to the subspace generated by low frequencies. Since the eigenvalues of the continuous operator appear in the denominator in formula (16), the relative error could be higher even if the absolute approximation is of high precision.

We address this issue in the next section.

For notational completeness, we stress that in the following the geometrical map G is always the identity and hence the matrices we consider are those denoted as $\mathbf{L}_n(a, b) := \mathbf{L}_n(a, b, I)$ with $\mathbf{L}_n(a, b, G)$ as in (6); furthermore, the regularity will be maximal that is $k = p - 1$ and hence $e_p(\theta)$, $f_p(\theta)$, $h_p(\theta)$ will denote $\mathbf{e}_{[p, p-1]}(\theta)$, $\mathbf{f}_{[p, p-1]}(\theta)$, $\mathbf{h}_{[p, p-1]}(\theta)$, respectively.

4 Global Distribution Results and Extrapolation

As discussed in [17], the mismatch between the analytical predictions and the eigenvalue errors essentially occurs only for small eigenvalues and a way to significantly reduce it has already been illustrated in that paper. Drawing inspiration from [1, 9, 10, 12, 13], we here describe—in the case of an isogeometric Galerkin discretization based on B-splines of the eigenvalue problem (1)—an alternative interpolation–extrapolation procedure to considerably improve the analytical predictions for small eigenvalues. We also illustrate the performance of this procedure by two examples.

1. We assume that there exists a function $c_p : [0, \pi] \rightarrow \mathbb{R}$, depending only on p and the coefficients a, b of the considered eigenvalue problem (1), such that, when using an isogeometric p -degree C^{p-1} B-spline discretization, the following property holds: independently of the parameter n , all the eigenvalues of the matrix $\frac{1}{n^2} \mathbf{L}_n(a, b)$, sorted in increasing order, satisfy

$$\frac{\lambda_{j,n}}{n^2} \approx c_p(\theta_{j,n}) e_p(\theta_{j,n}), \quad j = 1, \dots, \min(n + p - 2, n), \quad (17)$$

where

$$\begin{aligned} e_p &= \frac{f_p(\theta)}{h_p(\theta)} = \frac{(2 - 2 \cos \theta) h_{p-1}(\theta)}{h_p(\theta)} \\ &= \frac{(2 - 2 \cos \theta) (\phi_{[2p-1]}(p) + 2 \sum_{\ell=1}^{p-1} \phi_{[2p-1]}(p - \ell) \cos(\ell\theta))}{\phi_{[2p+1]}(p + 1) + 2 \sum_{\ell=1}^p \phi_{[2p+1]}(p + 1 - \ell) \cos(\ell\theta)} \end{aligned}$$

is defined in [17], and

$$\theta_{j,n} = \frac{j\pi}{n}, \quad j = 1, \dots, n.$$

2. Compute the eigenvalues of the matrix $\frac{1}{n_1^2} \mathbf{L}_{n_1}(a, b)$ corresponding to a small value of n_1 , i.e.,

$$\frac{\lambda_{j_1, n_1}}{n_1^2}, \quad j_1 = 1, \dots, n_1 + p - 2.$$

Since n_1 is small, this computation can be efficiently performed by any standard eigensolver (e.g., the MATLAB `eig` function).

3. Under the assumption in item 1,

$$c_p(\theta_{j_1, n_1}) \approx \frac{\lambda_{j_1, n_1}}{n_1^2 e_p(\theta_{j_1, n_1})}, \quad j_1 = 1, \dots, \min(n_1 + p - 2, n_1).$$

This means that we have an approximation of the unknown function c_p over the coarse uniform grid consisting of the points θ_{j_1, n_1} .

4. Interpolate the data

$$\left(\theta_{j_1, n_1}, \frac{\lambda_{j_1, n_1}}{n_1^2 e_p(\theta_{j_1, n_1})} \right), \quad j_1 = 1, \dots, \min(n_1 + p - 2, n_1),$$

by using, e.g., the MATLAB `interp1` function with the `'spline'` option, so as to obtain an approximation of c_p over the whole interval $[0, \pi]$. We call this approximation \tilde{c}_p . Note that, by construction,

$$\tilde{c}_p(\theta_{j_1, n_1}) = \frac{\lambda_{j_1, n_1}}{n_1^2 e_p(\theta_{j_1, n_1})}, \quad j_1 = 1, \dots, \min(n_1 + p - 2, n_1).$$

5. Given a large n , compute approximations of the numerical eigenvalues $\lambda_{j,n}$ by replacing c_p with \tilde{c}_p in (17):

$$\lambda_{j,n} \approx n^2 \tilde{c}_p(\theta_{j,n}) e_p(\theta_{j,n}), \quad j = 1, \dots, \min(n + p - 2, n).$$

6. Compute analytical predictions for the eigenvalue errors as follows:

$$\frac{\lambda_{j,n}}{\lambda_j} - 1 \approx \frac{n^2 \tilde{c}_p(\theta_{j,n}) e_p(\theta_{j,n})}{\lambda_j} - 1, \quad j = 1, \dots, \min(n + p - 2, n).$$

The next two examples show that the interpolation–extrapolation algorithm defined by items 1–6 may lead to analytical predictions of the eigenvalue errors which perform better (for small eigenvalues) than the analytical predictions proposed in (16).

Table 1 Example 1 [Linear C^0 B-spline discretization with $a(x) = 2 + 0.5x$ and $b(x) = 1$]: The five smallest eigenvalues and eigenvalue approximations, scaled by 10^2 , for $n = 200$, $n' = 1500$, $p' = 5$, and $r = 10000$

p	$\lambda_{j,n'}$	$\lambda_{j,n}$	$n^2 \zeta_r \left(\frac{j}{n} \right)$	$n^2 \tilde{c}_1(\theta_{j,n}) e_1(\theta_{j,n})$
1	0.221308917999822	0.221313437064828	0.225759339802366	0.221202202766985
	0.885445782386009	0.885517680038136	0.894411370606914	0.884899608283264
	1.992343949797817	1.992707562710680	2.006135775082463	1.991307942612523
	3.542008222229894	3.543157060383101	3.560895188355945	3.540749097918976
	5.534445409962895	5.537249919330117	5.559588944750918	5.533651310530473

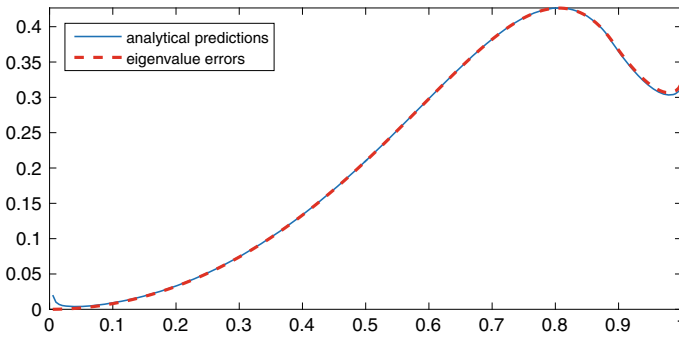


Fig. 1 Example 1 [Linear C^0 B-spline discretization with $a(x) = 2 + 0.5x$ and $b(x) = 1$]: analytical predictions $n^2 \zeta_r \left(\frac{j}{n} \right) / \lambda_{j,n'} - 1$ and eigenvalue errors $\lambda_{j,n} / \lambda_{j,n'} - 1$ versus j/N_n , $j = 1, \dots, N_n$ ($N_n = n - 1$, $n = 200$, $n' = 1500$, $p' = 5$, $r = 10000$)

Example 1 Let $p = 1$, $n = 200$, $a(x) = 2 + 0.5x$ and $b(x) = 1$. Let $n' = 1500 \gg n$, $p' = 5$, and take the first $n - 1$ eigenvalues of $\mathbf{L}_{n'}(a, b)$, namely $\lambda_{1,n'}, \dots, \lambda_{n-1,n'}$, as approximations of the unknown exact eigenvalues $\lambda_1, \dots, \lambda_{n-1}$. In Table 1 we report the five smallest eigenvalues and approximated eigenvalues for $p = 1$. As is clear the new approximation, $n^2 \tilde{c}_1(\theta_{j,n}) e_1(\theta_{j,n})$, performs better than $n^2 \zeta_r \left(\frac{j}{n} \right)$ to approximate $\lambda_{j,n}$, where the function ζ_r introduced in [17, Sect. 3.1] is defined in the following way. Sample $\frac{a(x)}{b(x)} e_1(\theta)$ at the grid points $(x, \theta) \in \mathcal{G}_r$, for a chosen r , where

$$\mathcal{G}_r = \left\{ \left(\frac{i}{r}, \frac{j\pi}{r} \right) : i, j = 1, \dots, r \right\}.$$

The samples are ordered in increasing order in a vector $(z_1, z_2, \dots, z_{r^2})$. Let $\zeta_r : [0, 1] \rightarrow \mathbb{R}$ be a piecewise linear non-decreasing function that interpolate the samples $(z_0 := z_1, z_2, \dots, z_{r^2})$ over the nodes $(0, \frac{1}{r^2}, \frac{2}{r^2}, \dots, 1)$.

In Fig. 1 we present the (approximate) analytical predictions $n^2 \zeta_r \left(\frac{j}{n} \right) / \lambda_{j,n'} - 1$, with $r = 10000$, together with the (approximate) eigenvalue errors $\lambda_{j,n} / \lambda_{j,n'} - 1$.

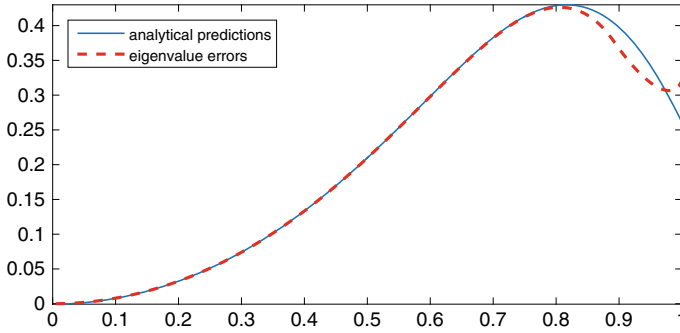


Fig. 2 Example 1 [Linear C^0 B-spline discretization with $a(x) = 2 + 0.5x$ and $b(x) = 1$]: analytical predictions $n^2 \tilde{c}_1(\theta_{j,n}) e_1(\theta_{j,n}) / \lambda_{j,n'} - 1$ and eigenvalue errors $\lambda_{j,n} / \lambda_{j,n'} - 1$ versus j/N_n , $j = 1, \dots, N_n$ ($N_n = n - 1$, $n = 200$, $n' = 1500$, $p' = 5$, $n_1 = 10$)

Table 2 Example 2 [p -degree C^{p-1} B-spline discretization for $p = 2, 3, 4, 5$ with $a(x) = 2.1 \cdot 10^9 + 1.05 \cdot 10^9 x$ and $b(x) = 8000$]: The five smallest eigenvalues and eigenvalue approximations, scaled by 10^7 , for $n = 200$, $n' = 1500$, $p' = 5$, and $r = 10000$

p	$\lambda_{j,n'}$	$\lambda_{j,n}$	$n^2 \zeta_{r,[p,p-1]} \left(\frac{j}{n} \right)$	$n^2 \tilde{c}_p(\theta_{j,n}) e_p(\theta_{j,n})$
2	0.320242073856735	0.320242073906214	0.326994616633570	0.319596822473787
	1.281965164064426	1.281965165832994	1.295209495587940	1.278576943997320
	2.884839278092422	2.884839298406320	2.904497248558355	2.877209798346387
	5.128863345308352	5.128863459872621	5.154973083201467	5.115745284850879
	8.014037230782240	8.014037668616439	8.046649299586466	7.994414445665448
3	0.320242073856735	0.320242073878847	0.326994616613888	0.319613348810547
	1.281965164064426	1.281965164062972	1.295209493290653	1.278636625771722
	2.884839278092422	2.884839278083705	2.904497230696836	2.877330529920427
	5.128863345308352	5.128863345316139	5.154972936807892	5.115937346593204
	8.014037230782240	8.014037230847546	8.046648648420712	7.994681508249463
4	0.320242073856735	0.320242073879743	0.326994616613781	0.319603273140503
	1.281965164064426	1.281965164063456	1.295209493290564	1.278600044346683
	2.884839278092422	2.884839278081923	2.904497230695864	2.877256125504316
	5.128863345308352	5.128863345304529	5.154972936792049	5.115818311034016
	8.014037230782240	8.014037230780470	8.046648648302227	7.994514965707993
5	0.320242073856735	0.320242073879344	0.326994616614055	0.319620210926112
	1.281965164064426	1.281965164062812	1.295209493290633	1.278661525969442
	2.884839278092422	2.884839278081491	2.904497230695859	2.877381146957867
	5.128863345308352	5.128863345303335	5.154972936792168	5.116018280099667
	8.014037230782240	8.014037230779344	8.046648648302266	7.994794677407651

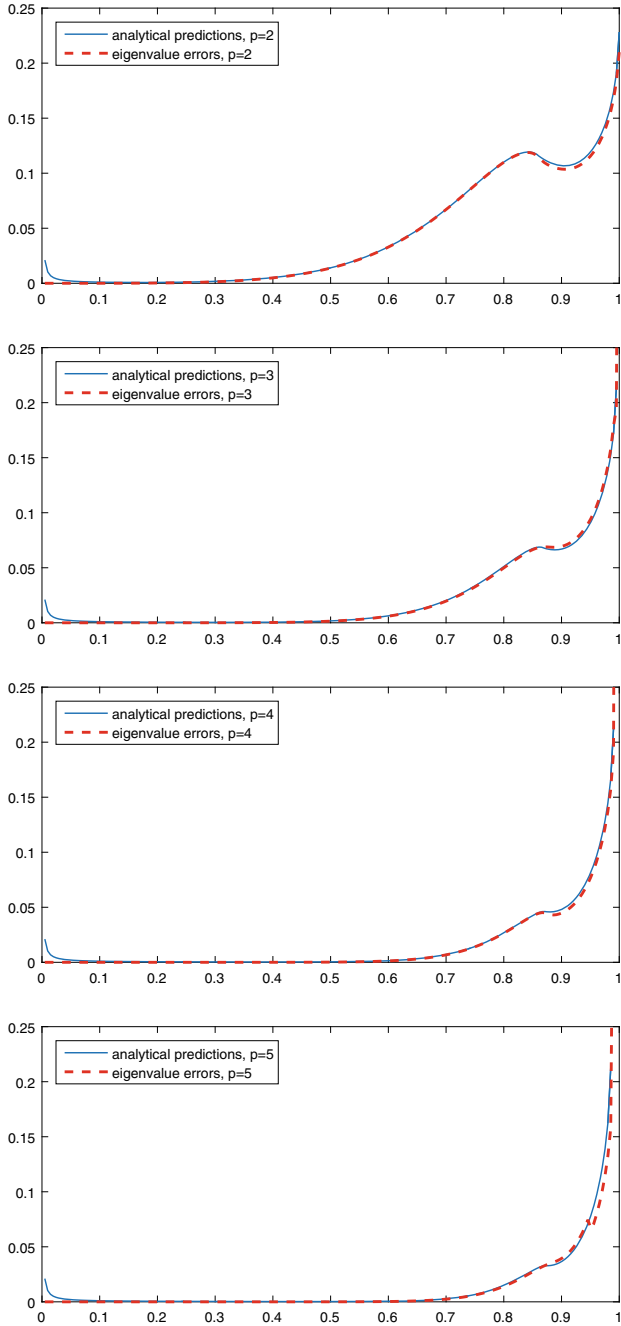


Fig. 3 Example 2 [p -degree C^{p-1} B-spline discretization for $p = 2, 3, 4, 5$ with $a(x) = 2.1 \cdot 10^9 + 1.05 \cdot 10^9 x$ and $b(x) = 8000$]; analytical predictions $n^2 \zeta_{r, [p, p-1]}(\frac{j}{n}) / \lambda_{j, n'} - 1$ and eigenvalue errors $\lambda_{j, n} / \lambda_{j, n'} - 1$ versus j/N_n , $j = 1, \dots, N_n$ ($N_n = n + p - 2$, $n = 200$, $n' = 1500$, $p' = 5$, $r = 10000$)

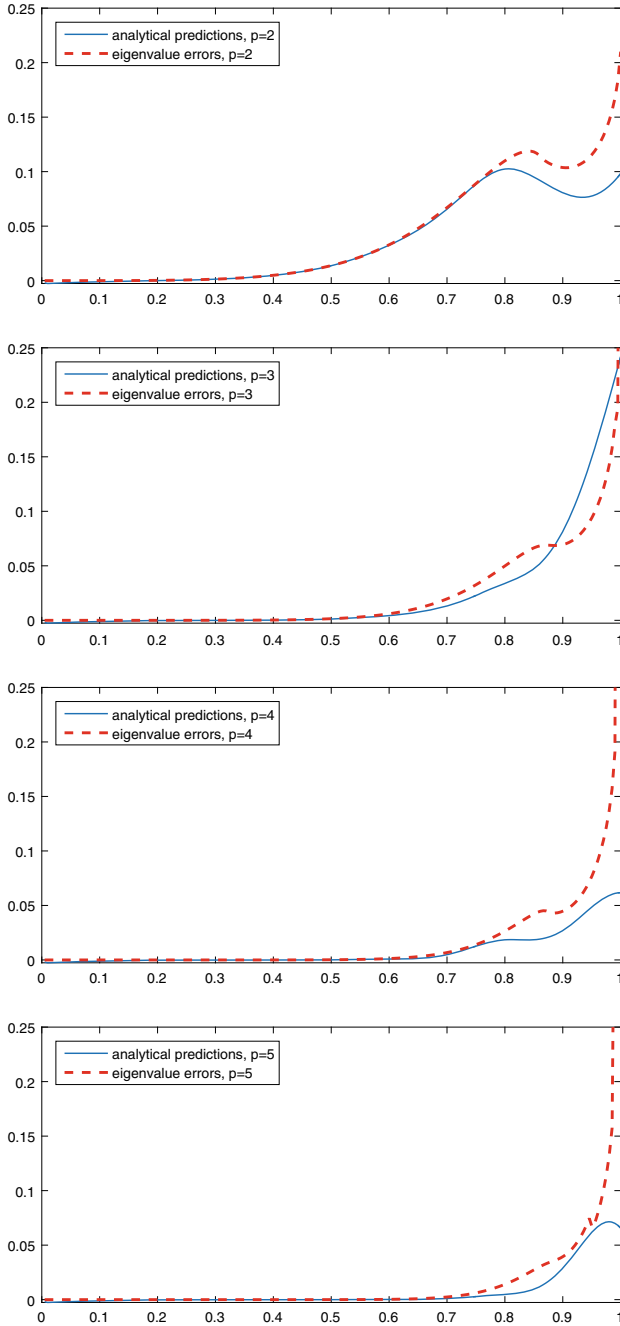


Fig. 4 Example 2 [p -degree C^{p-1} B-spline discretization for $p = 2, 3, 4, 5$ with $a(x) = 2.1 \cdot 10^9 + 1.05 \cdot 10^9 x$ and $b(x) = 8000$]: analytical predictions $n^2 \tilde{c}_p(\theta_{j,n}) e_p(\theta_{j,n}) / \lambda_{j,n'} - 1$ and eigenvalue errors $\lambda_{j,n} / \lambda_{j,n'} - 1$ versus j/N_n , $j = 1, \dots, N_n$ ($N_n = n + p - 2$, $n = 200$, $n' = 1500$, $p' = 5$, $n_1 = 10$)

We note the mismatch for the smallest eigenvalues. In Fig. 2 we plot the (approximate) analytical predictions $n^2 \tilde{c}_1(\theta_{j,n}) e_1(\theta_{j,n}) / \lambda_{j,n'} - 1$, obtained from the above interpolation–extrapolation algorithm for $n_1 = 10$, and the (approximate) eigenvalue errors $\lambda_{j,n} / \lambda_{j,n'} - 1$ versus $j / (n - 1)$, for $j = 1, \dots, n - 1$. We clearly see that in Fig. 2 the slight mismatch for small eigenvalues observed Fig. 1 has been reduced.

Example 2 Let $p = 2, 3, 4, 5, n = 200, a(x) = 2.1 \cdot 10^9 + 1.05 \cdot 10^9 x$ and $b(x) = 8000$. The approximation parameters n' and n_1 have been chosen as $n' = 1500$ and $n_1 = 10$, respectively, and the eigenvalues $\lambda_{j,n'}$ have been chosen correspondingly, in the sense that $\lambda_{j,n'}$ is computed as follows: we compute the eigenvalues $\lambda_{j,n'}$ for a matrix $\mathbf{L}_{n'}(a, b)$ of order $n' = 1500$ and B-spline degree $p' = 5$. Then we assume that the first $n + p - 2$ (for $n = 200$) of these eigenvalues, that is $j = 1, \dots, n + p - 2$, are accurate representations of the true eigenvalues λ_j ; see [17] for a motivation and a discussion on this matter.

In Table 2 we present the five smallest eigenvalues and approximated eigenvalues for $p = 2, 3, 4, 5$. As is clear the new approximation, $n^2 \tilde{c}_p(\theta_{j,n}) e_p(\theta_{j,n})$, performs better than $n^2 \zeta_{r,[p,p-1]} \left(\frac{j}{n}\right)$ to approximate $\lambda_{j,n}$. Figure 3 shows the comparison between the (approximate) analytical predictions

$$\frac{n^2 \zeta_{r,[p,p-1]} \left(\frac{j}{n}\right)}{\lambda_{j,n'}} - 1, \quad j = 1, \dots, n,$$

and the (approximate) eigenvalue errors

$$\frac{\lambda_{j,n}}{\lambda_{j,n'}} - 1, \quad j = 1, \dots, n + p - 2,$$

whereas Fig. 4 shows the comparison between the (approximate) analytical predictions

$$\frac{n^2 \tilde{c}_p(\theta_{j,n}) e_p(\theta_{j,n})}{\lambda_{j,n'}} - 1, \quad j = 1, \dots, n,$$

with the (approximate) eigenvalue errors. We see from Fig. 4 that the mismatch for small eigenvalues observed in Fig. 3 has been lowered.

5 Dyadic Decomposition Argument and Extreme Eigenvalues

While the distribution results are available both in the Galerkin [15] and in the collocation setting [7], the use of extrapolation methods, as those described in the previous section and in [10, 11], has been developed only in the Galerkin setting. The reason is the inherent nonsymmetry of the collocation matrices. However, this issue has to be

$$\tau_{n+1}(2 - 2 \cos \theta) = \begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Theorem 1 *There exists a constant $c_3 > 0$ such that*

$$\Re(\tilde{\mathbf{K}}_n^{[3]}) \geq c_3 \tau_{n+1}(2 - 2 \cos \theta), \quad \forall n \geq 2.$$

Proof For $n = 2, \dots, 5$, a direct verification shows that $\Re(\tilde{\mathbf{K}}_n^{[3]})$ is positive definite. Therefore, the theorem is proved if there exists a constant $c > 0$ such that

$$\Re(\tilde{\mathbf{K}}_n^{[3]}) \geq c \tau_{n+1}(2 - 2 \cos \theta), \quad \forall n \geq 6. \tag{21}$$

For $n \geq 6$ and $c > 0$, the matrix $\Re(\tilde{\mathbf{K}}_n^{[3]}) - c \tau_{n+1}(2 - 2 \cos \theta)$ is explicitly given by

$$\begin{bmatrix} \frac{11}{2} - 2c & -\frac{4}{3} + c & -\frac{1}{6} & & & & & & & & \\ -\frac{4}{3} + c & \frac{5}{2} - 2c & -1 + c & & & & & & & & \\ & -\frac{1}{6} & -1 + c & 2 - 2c & -1 + c & & & & & & \\ & & & -1 + c & 2 - 2c & -1 + c & & & & & \\ & & & & \ddots & \ddots & \ddots & & & & \\ & & & & & -1 + c & 2 - 2c & -1 + c & & & \\ & & & & & & -1 + c & 2 - 2c & -1 + c & -\frac{1}{6} & \\ & & & & & & & -1 + c & \frac{5}{2} - 2c & -\frac{4}{3} + c & \\ & & & & & & & & -\frac{1}{6} & -\frac{4}{3} + c & \frac{11}{2} - 2c \end{bmatrix}$$

and can be decomposed as follows:

$$\begin{bmatrix} r & -\frac{4}{3} + c \\ -\frac{4}{3} + c & s \end{bmatrix}_2^1 + \begin{bmatrix} \frac{11}{2} - 2c - r & 0 & -\frac{1}{6} \\ 0 & 0 & 0 \\ -\frac{1}{6} & 0 & t \end{bmatrix}_3^1 + \begin{bmatrix} \frac{5}{2} - 2c - s & -1 + c \\ -1 + c & 1 - c - t \end{bmatrix}_3^2 + \begin{bmatrix} 1 - c & -1 + c \\ -1 + c & 2 - 2c & -1 + c \\ & \ddots & \ddots & \ddots \\ & & -1 + c & 2 - 2c & -1 + c \\ & & & -1 + c & 1 - c \end{bmatrix}_{n-1}^3$$

$$+ \begin{bmatrix} 1 - c - t & -1 + c \\ -1 + c & \frac{5}{2} - 2c - s \end{bmatrix}_n^{n-1} + \begin{bmatrix} t & 0 & -\frac{1}{6} \\ 0 & 0 & 0 \\ -\frac{1}{6} & 0 & \frac{11}{2} - 2c - r \end{bmatrix}_{n+1}^{n-1} + \begin{bmatrix} s & -\frac{4}{3} + c \\ -\frac{4}{3} + c & r \end{bmatrix}_{n+1}^n .$$

In the above decomposition, $r, s,$ and t are arbitrary real numbers, whereas each term of the summation is a $(n + 1) \times (n + 1)$ matrix whose only nonzero entries are shown in the associated box and are contained in the principal submatrix corresponding to the rows from the superscript to the subscript. For instance,

$$\begin{bmatrix} \frac{11}{2} - 2c - r & 0 & -\frac{1}{6} \\ 0 & 0 & 0 \\ -\frac{1}{6} & 0 & t \end{bmatrix}_3^1 = \begin{bmatrix} \frac{11}{2} - 2c - r & 0 & -\frac{1}{6} & 0 & \dots & 0 \\ 0 & 0 & 0 & \vdots & \vdots & \\ -\frac{1}{6} & 0 & t & \vdots & \vdots & \\ 0 & \dots & \dots & 0 & \vdots & \\ \vdots & & & & \vdots & \\ 0 & \dots & \dots & \dots & 0 & \end{bmatrix} .$$

If each term of the above summation is a nonnegative definite matrix, then $\Re(\tilde{\mathbf{K}}_n^{[3]}) - c \tau_{n+1}(2 - 2 \cos \theta)$ is nonnegative definite as well. The following conditions ensure that each term of the summation is nonnegative definite:

$$\begin{cases} r > 0, s > 0, rs \geq (\frac{4}{3} - c)^2, \\ \frac{11}{2} - 2c - r > 0, t > 0, (\frac{11}{2} - 2c - r)t \geq \frac{1}{36}, \\ \frac{5}{2} - 2c - s > 0, 1 - c - t > 0, \\ (\frac{5}{2} - 2c - s)(1 - c - t) \geq (1 - c)^2, \\ c \leq 1. \end{cases}$$

These conditions are satisfied, for instance, with

$$c = \frac{58}{59}, s = \frac{1}{2}, r = \frac{(\frac{4}{3} - c)^2}{s}, t = \frac{\frac{1}{36}}{\frac{11}{2} - 2c - r} .$$

Hence, (21) holds with $c = 58/59$. □

We verified that the dyadic decomposition argument used in the proof of Theorem 1 can also be used to prove (20) for $p = 2, 4$. Although this argument becomes quite difficult to apply for $p \geq 5$, we have reason to believe that a careful application of it could prove (20) for any given $p \geq 2$.

The above theorem and the relations in (20) have interesting consequences on the conditioning measured with respect to the induced Euclidean norm as reported in Corollary 1, whose proof requires a technical but interesting in itself lemma.

Lemma 1 *Given X square matrix, assuming $\Re(X)$ positive definite, we have*

$$\lambda_{\min}(\Re(X)) \leq \sigma_{\min}(X).$$

Proof As a first step define the following matrix \hat{X} of order $2n$

$$\hat{X} = \begin{bmatrix} 0 & X \\ X^H & 0 \end{bmatrix}.$$

By taking into account the standard singular value decomposition, it is easy to see (and it is well-known) that the singular values $\sigma_j(X)$ of X , $j = 1, \dots, n$, coincide with the n largest eigenvalues of \hat{X} , since the eigenvalues of the Hermitian matrix \hat{X} are $\pm\sigma_j(X)$, $j = 1, \dots, n$. Therefore, we infer that

$$\sigma_{\min}(X) = \lambda_{n+1}(\hat{X}),$$

with $\lambda_1(\hat{X}) \leq \lambda_2(\hat{X}) \leq \dots \leq \lambda_{2n}(\hat{X})$. From the minimax characterization (see [4]) of the $(n + 1)$ -th eigenvalue of \hat{X} , we obtain

$$\sigma_{\min}(X) = \max_{\dim(U)=n} \min_{u \in U} \frac{u^H \hat{X} u}{u^H u}, \quad (22)$$

where the maximum is taken over all subspaces U with codimension n in \mathbb{C}^{2n} or, equivalently, over all subspaces U of dimension n . Let U_* be the n -dimensional subspace of \mathbb{C}^{2n} made of all vectors y of the form

$$y = \begin{pmatrix} u \\ u \end{pmatrix}, \quad u \in \mathbb{C}^n, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

If $y \in U_*$ is partitioned into blocks as above, it holds

$$\frac{y^H \hat{X} y}{y^H y} = \frac{u^H \Re(X) u}{u^H u}, \quad (23)$$

and since U_* is a particular subspace of dimension n , from (22) the thesis follows. \square

Corollary 1 *Assume that (20) is satisfied for a given p . Then*

$$\mu_2(\Re(\tilde{\mathbf{K}}_n^{[p]})) \sim n^2, \quad (24)$$

and

$$\mu_2(\tilde{\mathbf{K}}_n^{[p]}) \leq C_p n^2, \quad (25)$$

with C_p a positive constant depending only on p and $\mu(X) = \|X\| \|X^{-1}\|$ being the conditioning of an invertible matrix X measured with respect to the induced Euclidean norm $\|\cdot\|$.

Proof First of all we observe that both $\Re(\tilde{\mathbf{K}}_n^{[p]})$ and $\tilde{\mathbf{K}}_n^{[p]}$ are banded matrices with coefficients not depending on the parameter n . Hence we have that their induced Euclidean norm is asymptotic to a constant not depending on n that is

$$\|\Re(\tilde{\mathbf{K}}_n^{[p]})\| \sim 1, \quad (26)$$

$$\|\tilde{\mathbf{K}}_n^{[p]}\| \sim 1. \quad (27)$$

We first prove (24). By (20) we have

$$\Re(\tilde{\mathbf{K}}_n^{[p]}) \geq c_p \tau_{n+p-2}(2 - 2 \cos \theta), \quad \forall n \geq 2,$$

with $c_p > 0$ and hence $\Re(\tilde{\mathbf{K}}_n^{[p]})$ is positive definite and its minimal eigenvalue is bounded from below by

$$c_p 4 \sin^2 \left(\frac{\pi}{2(n+p-1)} \right),$$

$4 \sin^2 \left(\frac{\pi}{2(n+p-1)} \right)$ being the minimal eigenvalue of $\tau_{n+p-2}(2 - 2 \cos \theta)$ (see [19] and references therein). On the other hand, $\Re(\tilde{\mathbf{K}}_n^{[p]})$ contains as a principal submatrix a Toeplitz matrix of size proportional to n and generated by f_p being nonnegative and having a unique zero of order two (see [7, 14]). Hence such a Toeplitz matrix has the minimal eigenvalue l_n asymptotic to n^{-2} (see again [19] and references therein) and by the Cauchy interlacing theorem the minimal eigenvalue of $\Re(\tilde{\mathbf{K}}_n^{[p]})$ is bounded from above by $l_n \sim n^{-2}$. Therefore

$$\lambda_{\min}(\Re(\tilde{\mathbf{K}}_n^{[p]})) \sim n^{-2}.$$

But $\Re(\tilde{\mathbf{K}}_n^{[p]})$ is positive definite and hence

$$\left\| [\Re(\tilde{\mathbf{K}}_n^{[p]})]^{-1} \right\| = [\lambda_{\min}(\Re(\tilde{\mathbf{K}}_n^{[p]}))]^{-1} \sim n^2,$$

from which, using (26), we deduce

$$\mu_2(\Re(\tilde{\mathbf{K}}_n^{[p]})) \sim n^2.$$

We now prove (25). Given the known fact $\|X^{-1}\| = [\sigma_{\min}(X)]^{-1}$, statement (25) is a consequence of (27), of (20), of Lemma 1, and of the fact that the minimal eigenvalue of $\tau_{n+p-2}(2 - 2 \cos \theta)$ is

$$\sin^2\left(\frac{\pi}{2(n+p-1)}\right) \sim n^{-2}.$$

□

Furthermore, we have reasons to conjecture that the spectral conditioning of the collocation stiffness matrices grows asymptotically as n^2 , exactly as in the Galerkin setting [14], since $\tilde{\mathbf{K}}_n^{[p]}$ contains as a principal submatrix a Toeplitz matrix of size proportional to n and generated by f_p having a unique zero of order two (as $\Re(\tilde{\mathbf{K}}_n^{[p]})$, see the proof of Corollary 1).

We finally stress that (20) is also important in a multigrid context for proving the optimality of the related two-grid and multigrid techniques (see [2, 6, 8, 19]), by following the theory reported by Ruge and Stüben in [18].

6 Conclusions

In this note, we have considered the spectral analysis of large matrices coming from the isogeometric approximations based on B-splines of the eigenvalue problem

$$-(a(x)u'(x))' = \lambda b(x)u(x), \quad x \in (0, 1),$$

where $u(0)$ and $u(1)$ are given. In the collocation setting, we complemented global eigenvalue distribution results, available in the literature [7], with precise estimates for the extremal eigenvalues and hence for the spectral conditioning of the resulting matrices. In the Galerkin setting, we have designed an efficient matrix-less procedure (see [9]) that gives a highly accurate estimate of the all the eigenvalues, starting from the knowledge of the spectral GLT distribution symbol. Possible extensions include a more systematic treatment of the collocation case, both via the use of dyadic decomposition arguments and via the use of proper matrix-less extrapolation techniques. This last item is completely new and represents a real challenge, thus it will be the subject of a future investigation.

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